UNIFORMLY LEVI DEGENERATE CR
MANIFOLDS; THE 5 DIMENSIONAL CASE

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ABSTRACT. In this paper, we consider real hypersurfaces M in $\mathbb{C}^3$ (or more generally, 5-dimensional CR manifolds of hypersurface type) at uniformly Levi degenerate points, i.e. Levi degenerate points such that the rank of the Levi form is constant in a neighborhood. We also require the hypersurface to satisfy a certain second order nondegeneracy condition (called 2-nondegeneracy) at the point. For a real-analytic everywhere Levi degenerate hypersurface M in $\mathbb{C}^3$ which is not locally equivalent to a hypersurface of the form $\tilde{M} \times \mathbb{C}$, such points are dense on M.

Our first result is the construction, near any point $p_0 \in M$ satisfying the above conditions, of a principal bundle $P \to M$ and a $\mathbb{R}^{\dim P}$-valued 1-form $\underline{\omega}$, uniquely determined by the CR structure on M, which defines an absolute parallelism on P and has the following property: Let M and M' be two real-analytic hypersurfaces in $\mathbb{C}^3$ with distinguished points $p_0 \in M$, $p_0' \in M'$ and parallelized principal bundles $P, \underline{\omega}$, $P', \underline{\omega}'$, respectively. Then there exists a local biholomorphism $h \colon (\mathbb{C}^3, p_0) \to (\mathbb{C}^3, p_0')$ with $H(M) \subset M'$ if and only if there exists a real-analytic diffeomorphism $H \colon P \to P'$ with $H^*\underline{\omega}' = \underline{\omega}$. ($H$ is then the lift of h, i.e. $\pi' \circ H = \pi \circ h$ where π and π' denote the projections $\pi \colon P \to M$, $\pi' \colon P' \to M'$). This solves the biholomorphic equivalence problem for uniformly Levi degenerate hypersurfaces in $\mathbb{C}^3$ at 2-nondegenerate points in view of Cartan's solution of the equivalence problem for absolute parallelisms.

A basic example of a hypersurface of the type under consideration is the tube $\Gamma_{\mathbb{C}}$ over the light cone. Our second result is the characterization of $\Gamma_{\mathbb{C}}$ by vanishing curvature conditions in the spirit of the characterization of the unit sphere as the flat model for strongly pseudoconvex hypersurfaces in $\mathbb{C}^{n+1}$ in terms of the Cartan–Chern–Moser connection.

0. INTRODUCTION

0.1. A brief history. A fundamental problem in the study of real submanifolds in complex space is the *biholomorphic equivalence problem* which in its most general form asks for (intrinsic) conditions on two submanifolds $M, M' \subset \mathbb{C}^N$ at distinguished points $p_0 \in M$, $p_0' \in M'$ which guarantee that there exists a local biholomorphism $H \colon \mathbb{C}^N \to \mathbb{C}^N$ defined near p_0 such that $H(p_0) = p_0'$ and $H(M \cap U) = M' \cap U'$, for some open neighborhoods $U, U' \subset \mathbb{C}^N$ of p_0 and p_0' respectively. When M and M' are real-analytic, an equivalent formulation is to ask for a real-analytic local CR diffeomorphism $f \colon M \to M'$ defined near $p_0 \in M$ with $f(p_0) = p_0'$. (For standard definitions and results on real submanifolds in complex space and abstract CR structures, the reader is referred e.g. to [BER]).

The case where M and M' are real-analytic and Levi nondegenerate hypersurfaces was solved by Cartan [C1–2] in $\mathbb{C}^2$, and by Tanaka [T1–2] and Chern–Moser [CM] in $\mathbb{C}^N$, $N \geq 2$. The solution consists of producing a fiber bundle $Y \to M$, for any given Levi nondegenerate hypersurface $M \subset \mathbb{C}^N$, and 1-form $\underline{\omega}$ on Y, valued in $\mathbb{R}^{\dim Y}$, which at every $y \in Y$ gives an isomorphism between $T_y Y$ and $\mathbb{R}^{\dim Y}$ (an absolute parallelism or $\{1\}$-structure on Y; see e.g. [KN] or [K]) such that the following holds. If there exists a CR diffeomorphism $f \colon M \to M'$, then there exists a diffeomorphism $F \colon Y \to Y'$ (where corresponding objects for M' are denoted with ') such that $F^* \underline{\omega}' = \underline{\omega}$ and the following diagram commutes

$$
(0.1.1) \qquad
\begin{array}{ccc}
Y & \xrightarrow{\ F\ } & Y' \\
\pi \downarrow & & \downarrow \pi' \\
M & \xrightarrow{\ f\ } & M'.
\end{array}
$$

Conversely, if there exists a diffeomorphism $F \colon Y \to Y'$ such that $F^* \underline{\omega}' = \underline{\omega}$, then there exists a CR diffeomorphism $f \colon M \to M'$ such that (0.1.1) commutes. Suppose that such a bundle $Y \to M$ and $\mathbb{R}^{\dim Y}$-valued 1-form $\underline{\omega}$ can be constructed for every M in some given class of manifolds. Then we shall say that the bundle $Y \to M$ with 1-form $\underline{\omega}$ *reduces the CR structure on M to a parallelism* (in this class). The construction of a bundle $Y \to M$ which reduces the CR structure on M to a parallelism for a class of CR submanifolds in $\mathbb{C}^N$ reduces the biholomorphic equivalence problem for the real-analytic manifolds in this class to the equivalence problem for $\{1\}$-structures. The latter problem was solved by Cartan, and is well understood (see e.g. [G] or [S]).

The bundle $Y \to M$ constructed in [CM] is in fact a principal fiber bundle with group G_0, where G_0 is the isotropy subgroup of $SU(p+1, q+1)$, $p+q = N-1$, and p, q are the number of positive and negative eigenvalues, respectively, of the Levi form. The authors of [CM] also construct a Cartan connection Π valued in the Lie algebra $\mathfrak{su}(p+1, q+1)$ which defines the same parallelism as $\underline{\omega}$ (given a suitable identification of $\mathbb{R}^{\dim Y}$ with $\mathfrak{su}(p+1, q+1)$). Covariant differentiation of the curvature $\Omega := d\Pi - \Pi \wedge \Pi$ produces a complete set of invariants for a real-analytic Levi nondegenerate hypersurface. In particular, it follows that a real-analytic strongly pseudoconvex hypersurface in $\mathbb{C}^N$ is locally biholomorphic to a piece of the $(2N-1)$-sphere (the "standard model" for such hypersurfaces) if and only if the curvature Ω is identically zero (i.e. the connection is flat). The reader is also referred to the work of Burns-Shnider [BS] and Webster [We] for further discussion in the Levi nondegenerate case.

More recently, CR manifolds of higher codimension whose Levi forms are suitably nondegenerate have been studied by several authors. Since the main focus in the present paper is on hypersurfaces, we mention only the papers by Čap–Schichl [CS], Ezhov–Isaev–Schmalz [EIS], Garrity–Mizner [GM], Schmalz–Slovák [SS], and refer the interested reader to these papers for further information about the higher codimensional case.

In this paper, we consider real hypersurfaces (and, more generally, CR manifolds of hypersurface type) which have *degenerate* Levi forms. Before describing our main results, we should mention that another approach to the biholomorphic equivalence

were studied by the author in [E3–4]; another class of Levi degeneracies in $\mathbb{C}^2$ was considered by Wong [Wo]. However, at least to the best of the author's knowledge, the geometric approach as described above has not been previously studied for CR manifolds with degenerate Levi forms. The idea in this paper is to use the higher order invariant tensors introduced by the author in [E4] as a complement to the degenerate Levi form. We mostly restrict our attention to 5-dimensional manifolds in order to keep the number of cases and the notation to a minimum. The main results can be generalized to higher dimensional manifolds under Conditions 2.21 and 2.25; see the concluding remarks in §5.

The paper is organized as follows. Our main results are explained in §0.2. In §0.3, some examples of everywhere Levi degenerate hypersurfaces that arise e.g. in PDE theory are given. §1 is devoted to preliminary material including basic definitions and properties of Levi uniform CR manifolds. The necessary constructions for the main results are given in §2–3, and in §4 a discussion and characterization of the tube over the light cone is given. The paper concludes with some remarks in §5 about the higher dimensional case.

0.2. The main results. Our main results concern real hypersurfaces M in $\mathbb{C}^3$ or, more generally, 5-dimensional CR manifolds of hypersurface type, which are uniformly Levi degenerate in the sense that the Levi form has one nonzero and one zero eigenvalue in a neighborhood of a distinguished point $p_0 \in M$ (*Levi uniform* of rank 1 according to Definition 1.6 below). We also require M to be 2-nondegenerate at p_0 (see section 1 or [BER, Chapter XI]). The latter condition guarantees that if M is a real hypersurface in $\mathbb{C}^3$, then it is *holomorphically nondegenerate* (see [BER, Chapter XI]) and, in particular, not locally biholomorphic to a manifold of the form $\tilde{M} \times \mathbb{C}$ where $\tilde{M}$ is a hypersurface in $\mathbb{C}^2$. (However, M is always foliated by complex lines, but this foliation cannot be "straightened"; see Proposition 1.15.) In fact, if M is real-analytic and everywhere Levi degenerate, then at most points p (off a proper real-analytic subvariety) M is either locally biholomorphic to $\tilde{M} \times \mathbb{C}$ for some $\tilde{M} \subset \mathbb{C}^2$ or M is Levi uniform of rank 1 and 2-nondegenerate at p.

The most important example (indeed, the "standard model" for such manifolds) is the tube in $\mathbb{C}^3$ over the light cone in $\mathbb{R}^3$ (see Example 1.7 and section 4 for further discussion); other examples of everywhere Levi degenerate hypersurfaces that arise naturally are given, for motivation, in the next section. Let us just point out that the biholomorphically invariant geometry of the tube $\Gamma_\mathbb{C}$ over the light cone in $\mathbb{C}^4$ plays an important role in e.g. axiomatic quantum field theory since $\Gamma_\mathbb{C}$ bounds the so-called past and future tubes; see e.g. Sergeev–Vladimirov [SV] and Zhou [Z] and the references in these papers.

For the class of hypersurfaces described above, we define a new CR invariant $\hat{k}$ (see (2.34)); more generally, for $(2n+1)$-dimensional Levi uniform CR hypersurfaces of rank $n-1$ which satisfy Conditions 2.21 and 2.25, we introduce a sequence of invariants which in the 5-dimensional case reduce to the single invariant $\hat{k}$. One of our main results is the following. We refer the reader to section 1 for relevant definitions.

Theorem 1. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Then, there exists a principal fiber bundle $P \to M$ with group G_0 and a 1-form $\underline{\omega}$ on P which defines an*

on M to a parallelism. The group G_0 is a subgroup of $GL(\mathbb{R}^3)$ which has dimension two if the invariant $|\hat{k}(p_0)| = 2$ and dimension one otherwise.

Theorem 1 will be a consequence of the more detailed Theorems 3.1.37 and 3.2.9. We should mention that the group G_0, and hence the bundle P, in Theorem 1 is disconnected and has two components. In order to obtain a connected bundle, we have to choose an "orientation" for the Levi nullspace as explained §2 (see Theorems 3.1.37 and 3.2.9).

As mentioned above, the most important example is the tube $\Gamma_{\mathbb{C}}$ over the light cone for which the invariant $\hat{k} \equiv 2i$. We shall now characterize $\Gamma_{\mathbb{C}}$ among all M, as in Theorem 1 with $\hat{k} \equiv 2i$, by a curvature condition in the spirit of the characterization of the sphere among strongly pseudoconvex hypersurfaces as described section 0.1. There is a subgroup H of $GL(\mathbb{C}^4)$ and a subgroup H_0 of H such that H can be viewed as a principal fiber bundle over $\Gamma_{\mathbb{C}}$ with group H_0. The matrix valued Maurer-Cartan forms Π of H define a Cartan connection on $\Gamma_{\mathbb{C}}$ valued in $\mathfrak{h}$, the Lie algebra of H, with vanishing curvature $\Omega = d\Pi - \Pi \wedge \Pi$. (All this is explained in detail in section 4.) For a real-analytic CR manifold M as in Theorem 1 with the invariant $\hat{k} \equiv 2i$, we can identify the group G_0 with the group H_0, and construct, using $\underline{\omega}$, a $\mathfrak{h}$-valued 1-form Π which, unfortunately, in general is not a Cartan connection. However, we have the following result, which is a consequence of the more detailed Theorem 4.31.

Theorem 2. *Let M a real-analytic CR manifold satisfying the conditions in Theorem 1 with $\hat{k} \equiv 2i$. Then there exists a $\mathfrak{h}$-valued 1-form Π on the principal bundle $P \to M$, given by Theorem 1, which gives an isomorphism between $T_u P$ and $\mathfrak{h}$ for every $u \in P$ and with the following property. There exists a real-analytic CR diffeomorphism $f \colon M \to \Gamma_{\mathbb{C}}$, defined near $p_0 \in M$, if and only if the curvature*

$$(0.2.1) \qquad\qquad \Omega := d\Pi - \Pi \wedge \Pi$$

vanishes identically near p_0.

We conclude this section by giving an application of Theorem 1. Let $\mathrm{Aut}(M, p_0)$ denote the stability group of a CR manifold M at $p_0 \in M$, i.e. the group of germs at p_0 of local smooth CR diffeomorphisms $f \colon (M, p_0) \to (M, p_0)$. Suppose that M satisfies the conditions of Theorem 1 at p_0. Pick any point $u \in P_{p_0}$, where $P \to M$ is the principal G_0 bundle given by Theorem 1 and P_p denotes the fiber over $p \in M$. By [K, Theorem 3.2] and Theorem 1, the group $\mathrm{Aut}(M, p_0)$ embeds as a closed submanifold of the fiber $P_{p_0} \cong G_0$ via the mapping

$$(0.2.2) \qquad\qquad \mathrm{Aut}(M, p_0) \ni f \mapsto F(u) \in P_{p_0},$$

where $F \colon P \to P$ is the lift of f as in the diagram (0.1.1). Thus, $\dim \mathrm{Aut}(M, p_0)$ is at most 2 if the invariant $|\hat{k}(p_0)| = 2$ and at most 1 if $|\hat{k}(p_0)| \neq 2$. We formulate this as follows.

Corollary 3. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Then $\dim \mathrm{Aut}(M, p_0) \leq 2$.*

The bound in Corollary 3 cannot be improved, since $\dim \mathrm{Aut}(\Gamma_{\mathbb{C}}, p) = 2$ for any $p \in \Gamma_{\mathbb{C}}$ as is shown in section 4. We should mention that in the recent preprint [Er], it was shown that the bound $\mathrm{Aut}(M, p_0) \leq 3$ holds for the class of all real-analytic 2-nondegenerate hypersurfaces in $\mathbb{C}^3$; observe that when M is real-analytic then, by the reflection principle (see [BJT]), every $f \in \mathrm{Aut}(M, p_0)$ is real-analytic, since

0.3. Examples of everywhere Levi degenerate CR manifolds. Trivial examples of real hypersurfaces in $\mathbb{C}^{n+1}$ which are everywhere Levi degenerate can be obtained by taking any hypersurface of the form $\tilde{M} \times \mathbb{C}$, where $\tilde{M}$ is a real hypersurface in $\mathbb{C}^n$. Such hypersurfaces, as mentioned in the previous section, are never 2-nondegenerate and, hence, are not of interest to us in the present paper. We give here two (from our viewpoint) more interesting classes of everywhere Levi degenerate hypersurfaces in $\mathbb{C}^{n+1}$. The reader is referred to section 1 for relevant definitions.

Example 0.3.1 (Everywhere characteristic hypersurfaces). Let $p(x)$ be a homogeneous polynomial of $x = (x_1, \ldots, x_{n+1})$. A real hypersurface $M \subset \mathbb{C}^{n+1}$ is called *characteristic* at $p_0 \in M$ for the partial differential operator $p(\partial) := p(\partial/\partial Z)$ if

$$(0.3.2) \qquad\qquad p\left(\partial\rho(p_0, \bar{p}_0)/\partial Z\right) = 0,$$

where $\rho(Z, \bar{Z}) = 0$ is a defining equation for M near p_0 and

$$(0.3.3) \qquad\qquad \partial/\partial Z = (\partial/\partial Z_1, \ldots, \partial/\partial Z_{n+1}).$$

M is called *everywhere characteristic* if M is characteristic at every point. Everywhere characteristic hypersurfaces for a given operator $p(\partial)$ arise as natural boundaries for the holomorphic continuation of (holomorphic) solutions of $p(\partial)u = 0$ (see e.g. [Hö, Chapter IX.4]). A concrete example is given by the so-called *Lie ball* defined by the equation

$$(0.3.4) \qquad\qquad |Z|^2 + \left(|Z|^4 - \left|\sum_{k=1}^{n+1} Z_k^2\right|^2\right)^{1/2} < 1.$$

The Lie ball is the maximal domain in $\mathbb{C}^{n+1}$ to which every harmonic function in the unit ball of $\mathbb{R}^{n+1}$ can be holomorphically continued (see e.g. [A]; cf. also [E1]). The boundary of the Lie ball is everywhere characteristic (at every smooth point) for the "Laplace operator"

$$(0.3.5) \qquad\qquad \sum_{j=1}^{n+1} (\partial/\partial Z_k)^2.$$

Another example is the tube over the light cone (see Example 1.7) which is everywhere characteristic for the "wave operator"

$$(0.3.6) \qquad\qquad \sum_{j=1}^{n} (\partial/\partial Z_k)^2 - (\partial/\partial Z_{n+1})^2.$$

We have the following.

Proposition 0.3.7. *Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface which is everywhere characteristic for a homogeneous partial differential operator $p(\partial)$. Then M is everywhere Levi degenerate.*

Proof. Pick $p_0 \in M$ and let $\rho(Z, \bar{Z}) = 0$ be a defining equation for M near $p_0 \in M$. We first claim that the CR vector field

$$(0.3.8) \qquad\qquad L := \sum^{n+1} \overline{\partial p(\partial\rho/\partial Z)/\partial x_k} \frac{\partial}{\partial \bar{Z}_k}$$

is tangent to M. Indeed, since M is everywhere characteristic for $p(\partial)$, we have

$$(0.3.9) \qquad\qquad p\left(\partial\rho/\partial Z\right) = a\rho$$

for some function a. The claim now follows from Euler's formula. By differentiating (0.3.9), it is straightforward (and left to the reader) to verify that L is a nullvector for the Levi form at every $p \in M$ near p_0. This proves the proposition. $\square$

Example 0.3.10. Let $p(x)$ be a homogeneous polynomial in $x = (x_1, \ldots, x_{n+1})$ with real coefficients and assume that $\partial p/\partial x$ is not identically zero along the variety $V_{\mathbb{R}} := \{x \in \mathbb{R}^{n+1} : p(x) = 0\}$. Then the tube $V_{\mathbb{C}}$ over $V_{\mathbb{R}}$ in $\mathbb{C}^{n+1}$,

$$(0.3.11) \qquad\qquad V_{\mathbb{C}} := \{Z \in \mathbb{C}^{n+1} : p(\operatorname{Re} Z) = 0\},$$

is a real hypersurface (outside a lower dimensional real algebraic subvariety) which we denote by M. The "radial" CR vector field

$$(0.3.12) \qquad\qquad L = \sum_{j=1}^{n+1} \operatorname{Re} Z_j\, \partial/\partial\bar{Z}_j$$

is tangent to M, and the reader can easily verify that L is a null vector for the Levi form of M at every $p \in M$. A concrete example is again the tube over the light cone (Example 1.7). Another example is the cubic defined by (0.3.11) with $p(x) = x_1^3 + x_2^3 - x_3^3$ which was given by Freeman [F] as an example of a manifold foliated by complex curves but not locally biholomorphic to a manifold of the form $\tilde{M} \times \mathbb{C}$.

1. Preliminaries

Let M be a CR manifold with CR bundle $\mathcal{V}$. Recall that this means that $\mathcal{V}$ is a subbundle of the compexified tangent bundle $\mathbb{C}TM$ such that $\mathcal{V}_p \cap \bar{\mathcal{V}}_p = \{0\}$ for every $p \in M$, and $\mathcal{V}$ is *formally integrable* i.e. any commutator between sections of $\mathcal{V}$ is again a section of $\mathcal{V}$; sections of $\mathcal{V}$ will henceforth be called *CR vector fields*. We shall denote the CR dimension of M, i.e. the (complex) dimension of the fibers $\mathcal{V}_p$ for $p \in M$, by n. We shall assume that M is of *hypersurface type*, i.e. the complex dimension of $T_p^0 M := (\mathcal{V}_p \oplus \bar{\mathcal{V}}_p)^{\perp} \subset \mathbb{C}T_p^* M$, for $p \in M$, is one. In particular, the dimension of M is $2n + 1$. For the remainder of this paper unless explicitly stated otherwise, all CR manifolds will be of hypersurface type. The bundle $T^0 M \subset \mathbb{C}T^* M$ is called the *characteristic bundle*, and real sections of $T^0 M$ are called *characteristic forms*. The subbundle $T' M \subset \mathbb{C}T^* M$, defined at $p \in M$ by $T_p' M := \mathcal{V}_p^{\perp}$, is called the *holomorphic cotangent bundle*. The formal integrability of $\mathcal{V}$ is equivalent to the following property of $T' M$: If ω is a section of $T' M$, then $d\omega$ is a section of the ideal generated by $T' M$ in the exterior algebra of $\mathbb{C}TM$.

Let $L_{\bar{1}}, \ldots, L_{\bar{n}}$ be a basis for the CR vector fields near some distinguished point $p_0 \in M$. Also, let θ be a nonvanishing characteristic form near the same point p_0. Following [E2] (see also [BER] and [E4]), we define linear operator $\mathcal{T}_{\bar{1}}, \ldots \mathcal{T}_{\bar{n}}$ on the holomorphic 1-forms on M, i.e. the sections of $T' M$, near p_0 as follows

$$(1.1) \qquad\qquad \mathcal{T}_{\bar{\alpha}\omega} := \frac{1}{\cdots} L_{\bar{\alpha}} \lrcorner\, d\omega.$$

where $\lrcorner$ denotes the usual contraction by a vector field. For $p \in M$ near p_0 and positive integers k, we define the subspace $E_{k,p} \subset T'_p M$ as the (complex) linear span of θ_p and $(T_{\bar{A}_j} \ldots T_{\bar{A}_1}\theta)_p$, for all $1 \leq j \leq k$ and all j-tuples $(A_1, \ldots, A_j) \in \{1, \ldots, n\}^j$. We define $E_{0,p}$ to be $T_p^0 M$. The CR manifold M is said to be *finitely nondegenerate* at p of

$$(1.2) \qquad E_{k,0} = T'_p M$$

for some integer $k \geq 1$, and k_0-nondegenerate at p if k_0 is the smallest integer k for which (1.2) holds. It was shown in [E2] (see also [BER, Chapter XI]) that this definition is consistent with the one for real hypersurfaces of $\mathbb{C}^{n+1}$ given in [BHR]. (These notions can also be extended to CR manifolds of arbitrary codimension; see [BER] or [E4].) For each integer k such that $E_k(p) \neq T'_p M$, the author introduced in [E4] an invariant tensor

$$(1.3) \qquad \psi_{k+1} \in \mathcal{V}_p^* \otimes \ldots \mathcal{V}_p^* \otimes F_{k,p}^* \otimes (T_p^0 M)^*,$$

where $\mathcal{V}_p^*$ occurs k times in (1.3) and $F_{k,p} = E_{k,p}^\perp \cap \bar{\mathcal{V}}_p$. The sequence of tensors $\psi_2, \ldots, \psi_{k_0+1}$ describes in more detail the data associated with k_0-nondegeneracy. In this paper, we shall mainly consider certain classes of 2-nondegenerate CR manifolds and, hence, we shall only be interested in the first two tensors ψ_2 and ψ_3. The reader is referred to [E4] for the precise definition of the tensors ψ_k and their basic properties.

The second order tensor ψ_2 at a point $p \in M$ is given in the bases $L_{\bar{1},p}, \ldots, L_{\bar{n},p}$ of $\mathcal{V}_p$ and θ_p of $T_p^0 M$ as an $n \times n$ matrix $(g_{\bar{A}B}(p))_{1 \leq A,B, \leq n}$, where

$$(1.4) \qquad g_{\bar{A}B}(p) := \langle (T_{\bar{A}}\theta)_p, L_{B,p} \rangle .$$

We use here, and throughout this paper, the convention that $L_B = \overline{L_{\bar{B}}}$. The reader should observe that we have the identity

$$(1.5) \qquad g_{\bar{A}B}(p) = \frac{1}{2i} \langle \theta_p, [L_B, L_{\bar{A}}]_p \rangle ,$$

which coincides with the *Levi form* $\mathcal{L}_\theta$ of M at p and θ_p. From this observation, we see that 1-nondegeneracy of M at p is equivalent to the classical notion of *Levi nondegeneracy* of M at p. Moreover, the subspace $F_{1,p} \subset \bar{\mathcal{V}}_p$ coincides with the *Levi nullspace* of the Levi form, i.e. those vectors $X_p \in \bar{\mathcal{V}}_p$ for which the linear form $Y_p \mapsto \mathcal{L}_\theta(Y_p, X_p)$ is zero. For the remainder of this paper, we shall use the notation $\mathfrak{N}_p$ for the Levi nullspace $F_{1,p}$.

Definition 1.6. A CR manifold M of hypersurface type is *1-uniform* or *Levi uniform* (of rank r) at $p \in M$ if the rank of the Levi form is constant (and equal to r) in a neighborhood of p.

Observe that the extremal cases of CR manifolds which are Levi uniform of rank 0 or n at a point p are precisely those which are Levi flat or Levi nondegenerate, respectively, at p. Such CR manifolds which, in addition, are real-analytic are by now fairly well understood: a real-analytic Levi flat CR manifold is locally CR equivalent to the real hyperplane $\operatorname{Im} Z_{n+1} = 0$ in $\mathbb{C}^{n+1}$, and a theory for real-analytic Levi nondegenerate CR manifolds was developed by E. Cartan [C1-2], Tanaka [T1-2], and Chern–Moser [CM].

The reader should also observe that any real-analytic CR manifold is Levi uniform outside a proper real-analytic subvariety (in particular, on a dense open subset). Before proceeding, let us pause and give an example of a Levi uniform CR

Example 1.7. The tube in $\mathbb{C}^3$ over the light cone in $\mathbb{R}^3$, i.e. the variety defined by

$$(1.8) \qquad\qquad (\mathrm{Re}\ Z_1)^2 + (\mathrm{Re}\ Z_2)^2 - (\mathrm{Re}\ Z^3)^2 = 0,$$

is Levi uniform of rank 1 at every nonsingular point, i.e. at every point where it is a real submanifold. Thus, it is Levi uniform, but neither Levi flat nor Levi nondegenerate since the CR dimension n is 2. The reader can also verify that the CR manifold given by equation (1.8) is 2-nondegenerate at every nonsingular point. This example will be discussed in greater detail in §4 below. (See also [E3], where this example is further discussed in connection with a normal form for 2-nondegenerate hypersurfaces in $\mathbb{C}^3$.)

Note that if M is Levi uniform (of rank r) at $p_0 \in M$, then the subspaces $\mathfrak{N}_p$ for p near p_0 form a (rank $n - r$) subbundle $\mathfrak{N}$ of $\bar{\mathcal{V}}$. From now on, we assume that M is Levi uniform of rank r, with $0 < r < n$, in a neighborhood of p_0 (to which we restrict our attention). We may arrange our basis for the CR vector fields $L_{\bar{1}}, \dots, L_{\bar{n}}$ so that $L_{r+1}, \dots, L_n$ is a basis for the sections of $\mathfrak{N}$ near that point. The second order tensor for p near p_0 then takes the form

$$(1.9) \qquad\qquad (g_{\bar{A}B}) = \begin{pmatrix} g_{\bar{\alpha}\beta} & 0 \\ 0 & 0 \end{pmatrix},$$

where $(g_{\bar{\alpha}\beta})_{1 \leq \alpha,\beta \leq r}$ is an $r \times r$ nondegenerate Hermitian matrix of smooth functions. In what follows, we shall use the summation convention and also the convention that capital roman indices $A, B, \dots$ run over the integers $\{1, \dots, n\}$ and Greek indices $\alpha, \beta, \dots$ run over $\{1, \dots, r\}$. The third order tensor ψ_3 can be represented near p_0 by $n \times n$ matrices $(h_{\bar{A}\bar{B}k})_{1 \leq A,B \leq n}$, $k = r+1, \dots, n$, of smooth functions, where

$$(1.10) \qquad\qquad h_{\bar{A}\bar{B}k} := \langle \mathcal{T}_{\bar{B}} \mathcal{T}_{\bar{A}} \theta, L_k \rangle.$$

Proposition 1.11. *Assume that M is Levi uniform of rank r at p_0. Then, in the notation introduced above, for every $k = r+1, \dots, n$ and all p in a neighborhood of p_0, it holds that $h_{\bar{A}\bar{B}k} = 0$ whenever A or B belongs to $\{r+1, \dots, n\}$.*

Proof. Using a well known identity (see e.g. [He, Chapter I.2]; see also the remark concerning our normalization of the pairing $\langle \cdot, \cdot \rangle$ in [E4]), we have

$$
\begin{aligned}
(1.12) \qquad h_{\bar{A}\bar{B}k} :=\ & \langle \mathcal{T}_{\bar{B}} \mathcal{T}_{\bar{A}} \theta, L_k \rangle = \langle d(\mathcal{T}_{\bar{A}} \theta), L_{\bar{B}} \wedge L_k \rangle \\
=\ & L_{\bar{B}}(\langle \mathcal{T}_{\bar{A}} \theta, L_k \rangle) - L_k(\langle \mathcal{T}_{\bar{A}} \theta, L_{\bar{B}} \rangle) - \langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{B}}, L_k] \rangle \\
=\ & - \langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{B}}, L_k] \rangle,
\end{aligned}
$$

where the last identity follows from the facts that $\langle \mathcal{T}_{\bar{A}} \theta, L_k \rangle \equiv 0$ and $\langle \mathcal{T}_{\bar{A}} \theta, L_{\bar{B}} \rangle \equiv 0$. It is proved in [E4] that the matrices $h_{\bar{A}\bar{B}k}$ are symmetric, so to prove the proposition it suffices to show $h_{\bar{A}\bar{l}k} = 0$ in a neighborhood of p_0 for $k, l \geq r+1$, i.e. $\langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{l}}, L_k] \rangle = 0$ in view of (1.12). To this end note, using the fact that $L_{k,p}$ and $L_{l,p}$ are null vectors for the Levi form at every p near p_0, that

(1.13)

Thus, by also using the Jacobi identity, we obtain

$$(1.14) \qquad \langle \mathcal{T}_{\bar{A}}\theta, [L_{\bar{l}}, L_k] \rangle = \langle \theta, [L_{\bar{l}}, [L_k, L_{\bar{A}}]] \rangle + \langle \theta, [L_k, [L_{\bar{A}}, L_{\bar{l}}]] \rangle.$$

The second term on the right hand side of (1.14) vanishes since $[L_{\bar{A}}, L_{\bar{l}}]$ is a CR vector field by the formal integrability of $\mathcal{V}$ and L_k is a null vector field for the Levi form. To show that the first term also vanishes, we must show that $[L_k, L_{\bar{A}}]$ is a section of $\mathcal{V} \oplus \bar{\mathcal{V}}$. This fact follows again from the fact that L_k is a null vector field for the Levi form since the latter is equivalent to $\langle \theta, [L_k, L_{\bar{A}}] \rangle = 0$ for every $A = 1, \dots, n$. The proof of Proposition 1.11 is complete. $\square$

Let us digress briefly to note the following result which, although of no importance for the remainder of this paper, follows from (the proof of) Proposition 1.11 above.

Proposition 1.15. *If $M \subset \mathbb{C}^{n+1}$ is a real hypersurface which is Levi uniform of rank $r < n$ at $p_0 \in M$, then M is foliated by complex manifolds of dimension $n - r$ in a neighborhood of p_0.*

Remarks 1.16.
 (i) The following partial converse to Proposition 1.15 is easy to verify: if M is foliated by complex manifolds of dimension $n - r$ then the rank of the Levi form at any point is $\leq r$.
 (ii) We should point out that the foliation given by Proposition 1.15, even when M is real-analytic, can in general not be "straightened", i.e. it is in general not true that M, as in the proposition, is CR equivalent to a real hypersurface of the form $\tilde{M} \times \mathbb{C}^r \subset \mathbb{C}^{n+1}$, where $\tilde{M}$ is a real hypersurface in $\mathbb{C}^{n+1-r}$. Indeed, if M is CR equivalent to $\tilde{M} \times \mathbb{C}^r$ (which is a *holomorphically degenerate* hypersurface; see e.g. [BER, Chapter XI]), then it cannot be finitely nondegenerate at any point. Thus, the foliation of the tube over the light cone (Example 1.7) cannot be straightened.

Proof of Proposition 1.15. An immediate consequence of Proposition 1.11 and (1.12) is that, for $k, l \geq r + 1$,

$$(1.17) \qquad [L_k, L_{\bar{l}}] = \sum_{m=r+1}^{n} (a_{k\bar{l}}^m L_m + b_{k\bar{l}}^{\bar{m}} L_{\bar{m}}),$$

where the $a_{k\bar{l}}^m$ and $b_{k\bar{l}}^{\bar{m}}$ are smooth functions satisfying $a_{l\bar{k}}^m + \overline{b_{k\bar{l}}^{\bar{m}}} = 0$. Thus, by the Frobenius theorem, M is foliated near p_0 by $2(n-r)$-dimensional integral manifolds of $\mathrm{Re}\, L_k, \mathrm{Im}\, L_k, k = r+1, \dots, n$. By the Newlander–Nirenberg theorem and (1.17), these manifolds are $(n - r)$-dimensional complex submanifolds in $\mathbb{C}^{n+1}$. $\square$

Returning to the third order tensor ψ_3, we observe that Proposition 1.11 shows that the matrices $(h_{\bar{A}\bar{B}k})$ representing ψ_3 are of the form

$$(1.18) \qquad (h_{\bar{A}\bar{B}k}) = \begin{pmatrix} h_{\bar{\alpha}\bar{\beta}k} & 0 \\ 0 & 0 \end{pmatrix}.$$

We conclude this section with the following observation, whose proof is immediate and left to the reader, characterizing 2-nondegeneracy for Levi uniform CR manifolds of rank r in terms of the $r \times r$ matrices $(h_{\bar{\alpha}\bar{\beta}k})$ in (1.18).

Proposition 1.19. *Assume that M is Levi uniform of rank r at p_0. Then, in the notation introduced above, M is 2-nondegenerate at p_0 if and only if the symmetric matrices $(h_{\bar{\alpha}\bar{\beta}k}(p_0))_{1 \leq \alpha, \beta \leq r}, k = r + 1, \dots, n$, are linearly independent over $\mathbb{C}$.*

2. A G-structure for Levi uniform CR manifolds of rank $n-1$

We keep the notation from the previous section. However, from now on, we shall restrict ourselves to the case $r = n-1$, i.e. the case where the rank of the Levi form near p_0 is $n-1$. Thus, we assume that M is a smooth CR manifold (of hypersurface type) of CR dimension n which is Levi uniform of rank $n - 1$ at a distinguished point $p_0 \in M$. We shall restrict our attention to a small neighborhood of p_0. In what follows, M will denote a sufficiently small neighborhood of p_0. We have the following invariant subbundles of the cotangent bundle $\mathbb{C}T^*M$

$$(2.1) \qquad T^0M \subset T''M \subset T'M,$$

where T^0M and $T'M$ were introduced in §1 and where $T''M$ is defined by

$$(2.2) \qquad T''_p M = \{\omega_p \in T'_p M \colon \langle \omega_p, L_p \rangle = 0, \ \forall L_p \in \mathfrak{N}_p\}.$$

Observe that

$$(2.3) \qquad \dim T^0_p M = 1, \ \dim T''_p M = n, \ \dim T'_p M = n+1.$$

Let $\theta, \theta^1, \ldots, \theta^n$ be a basis for the holomorphic 1-forms (i.e. sections of $T'M$) with the additional properties that θ is real and a basis for the sections of T^0M and $\theta, \theta^1, \ldots, \theta^{n-1}$ is a basis for the sections of $T''M$. Any other such basis $\tilde{\theta}, \tilde{\theta}^1, \ldots, \tilde{\theta}^n$ is related to $\theta, \theta^1, \ldots, \theta^n$ by

$$(2.4) \qquad \begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^\alpha \\ \tilde{\theta}^n \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ \xi^\alpha & \xi^\alpha_\beta & \xi \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^n \end{pmatrix},$$

where the coefficients in the $(n+1) \times (n+1)$ matrix in (2.4) are smooth functions (all complex valued, except u which is real valued); also, recall that we are using the summation convention, and Greek indices run over the set $\{1, \ldots, n-1\}$ since $r = 1$ here.

Observe that $T''_p M \cap \overline{T''_p M} = T^0_p M$ and $T''M \cup \overline{T''M}$ is a rank $2n-1$ subbundle of $\mathbb{C}T^*M$. The 1-forms $\theta, \theta^\alpha, \theta^{\bar\alpha}$, where $\theta^{\bar\alpha} = \overline{\theta^\alpha}$ as mentioned in §1, yield a coframe for bundle $T''M \cup \overline{T''M}$. Consider the bundle $Y \to M$ consisting of all such coframes $(\omega, \omega^\alpha, \omega^{\bar\alpha})^\tau$,

$$(2.5) \qquad \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar\alpha} \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ \overline{u^\alpha} & 0 & \overline{u^\alpha_\beta} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^{\bar\beta} \end{pmatrix},$$

where $(u, u^\alpha, u^\alpha_\beta) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{C}^{n-1} \times GL(\mathbb{C}^{n-1})$. If we let $G \subset GL(\mathbb{C}^{2n-1})$ denote the group consisting of matrices of the form

$$(2.6) \qquad S = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ \overline{u^\alpha} & 0 & \overline{u^\alpha_\beta} \end{pmatrix}, \quad (u, u^\alpha, u^\alpha_\beta) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{C}^{n-1} \times GL(\mathbb{C}^{n-1}),$$

then $Y \to M$ is a principal fiber bundle over M with group G; (2.5) gives a trivialization of Y in which $(u, u^\alpha, u^\alpha_\beta)$, or S given by (2.6), are (global) coordinates of Y. We denote by $\mathfrak{g}$ the Lie algebra of G, i.e. the space of matrices

$$(2.7) \qquad T = \begin{pmatrix} v & 0 & 0 \\ v^\alpha & v^\alpha_\beta & 0 \\ \overline{v^\alpha} & 0 & \overline{v^\alpha_\beta} \end{pmatrix}, \quad (v, v^\alpha, v^\alpha_\beta) \in \mathbb{R} \times \mathbb{C}^{n-1} \times \mathcal{M}(\mathbb{C}^{n-1}),$$

where $\mathcal{M}(\mathbb{C}^{n-1})$ denotes the space of all $(n-1) \times (n-1)$ matrices. If we pull back the forms $\theta, \theta^A, \theta^{\bar{A}}$ (where capital Roman indices run over $\{1, \dots, n\}$) to Y, still denoting the pulled-back forms by $\theta, \theta^A, \theta^{\bar{A}}$, then (2.5) defines 1-forms $\omega, \omega^\alpha, \omega^{\bar{\alpha}}$ on Y. The reader can verify that the latter 1-forms are invariantly defined on Y, i.e. independent of the initial choice of $\theta, \theta^\alpha, \theta^{\bar{\alpha}}$ above.

Differentiating the 1-forms $\omega, \omega^\alpha, \omega^{\bar{\alpha}}$ we obtain

$$(2.8) \qquad d \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} = dSS^{-1} \wedge \begin{pmatrix} \omega \\ \omega^\beta \\ \omega^{\bar{\beta}} \end{pmatrix} + Sd \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^{\bar{\beta}} \end{pmatrix},$$

where $S \in G$ is given by (2.6). The elements of the matrix valued 1-form dSS^{-1} are Maurer–Cartan forms for the Lie group G (see e.g. [G]). Let $L, L_A, L_{\bar{A}}$ denote a dual basis relative to $\theta, \theta^A, \theta^{\bar{A}}$. Thus, the $L_{\bar{A}}$ form a basis for the CR vector fields, $L_A = \overline{L_{\bar{A}}}$, and L_n spans the Levi nullbundle $\mathfrak{N}$. We shall use the notation introduced in §1 for the second (the Levi form) and third order tensors (relative to the bases chosen). Since M is Levi uniform of rank $n-1$ near p_0, the Levi form $(g_{\bar{A}B})$ satisfies (1.9) with $r = n-1$, and the third order tensor $(h_{\bar{A}\bar{B}n})$ satisfies, by Proposition 1.11, $h_{\bar{n}\bar{B}n} = h_{\bar{A}\bar{n}n} = 0$ in a neighborhood of p_0. Moreover, the matrix $(h_{\bar{\alpha}\bar{\beta}n})$ is symmetric. $(g_{\bar{\alpha}\beta})$ is an invertible Hermitian matrix and we shall denote its inverse by $(g^{\alpha\beta})$, i.e. $g_{\bar{\alpha}\beta}g^{\bar{\alpha}\gamma} = \delta^\gamma_\beta$. Using the formal integrability of $\mathcal{V}$ and the fact that θ is real, we obtain

$$(2.9) \qquad d\theta = ig_{\bar{\alpha}\beta}\theta^{\bar{\alpha}} \wedge \theta^\beta + \phi \wedge \theta,$$

where ϕ is a real 1-form. Similarly by the formal integrability, we can write

$$(2.11) \qquad d\theta^\alpha = \eta^\alpha \wedge \theta + \eta^\alpha_\beta \wedge \theta^\beta + h^\alpha_{\bar{B}}\theta^{\bar{B}} \wedge \theta^n,$$

for some 1-forms η^α and η^α_β, and some matrix $(h^\alpha_{\bar{B}})$.

Lemma 2.12. *In the notation introduced above, we have*

$$(2.13) \qquad h^\beta_{\bar{n}} = 0, \quad 2ig^{\bar{\alpha}\beta}h_{\bar{\alpha}\bar{\gamma}n} = h^\beta_{\bar{\gamma}}.$$

Proof. Recall the operators $\mathcal{T}_{\bar{A}}$ introduced in §1. Observe that, by the definition of $T''M$ and the fact that L_n spans $\mathfrak{N}$, the 1-forms θ and $\mathcal{T}_{\bar{\alpha}}\theta$ form a basis for the sections of $T''M$, and $\mathcal{T}_{\bar{n}}\theta = c\theta$ for some smooth function c. Indeed, we have

(2.14)

for some smooth functions c^β, as is straightforward to verify. In view of (2.14), a direct calculation using the definition of the third order tensor shows

$$(2.15) \qquad \langle L_{\bar{B}} \lrcorner d\theta^\beta, L_n \rangle = 2i g^{\bar{\alpha}\beta} h_{\bar{\alpha}\bar{B}n}.$$

On the other hand, a calculation using (2.11) shows

$$(2.16) \qquad \langle L_{\bar{B}} \lrcorner d\theta^\beta, L_n \rangle = h^\beta_{\bar{B}},$$

which completes the proof. $\square$

Thus, we rewrite (2.11) as follows

$$(2.17) \qquad d\theta^\alpha = \eta^\alpha \wedge \theta + \eta^\alpha_\beta \wedge \theta^\beta + h^\alpha_{\bar{\beta}} \theta^{\bar{\beta}} \wedge \theta^n, \quad 2i g^{\bar{\nu}\alpha} h_{\bar{\nu}\bar{\beta}n} = h^\alpha_{\bar{\beta}}.$$

In view of (2.8), we can write

$$(2.18) \qquad d \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} \Delta & 0 & 0 \\ \Delta^\alpha & \Delta^\alpha_\beta & 0 \\ \Delta^{\bar{\alpha}} & 0 & \Delta^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\beta \\ \omega^{\bar{\beta}} \end{pmatrix} + \begin{pmatrix} i\hat{g}_{\bar{\mu}\nu} \omega^{\bar{\mu}} \wedge \omega^\nu \\ \hat{h}^\alpha_{\bar{\mu}} \omega^{\bar{\mu}} \wedge \theta^n \\ \overline{\hat{h}^\alpha_{\bar{\mu}}} \omega^\mu \wedge \theta^{\bar{n}} \end{pmatrix},$$

where $\Delta, \Delta^\alpha, \Delta^{\bar{\alpha}} := \overline{\Delta^\alpha}, \Delta^\alpha_\beta, \Delta^{\bar{\alpha}}_{\bar{\beta}} := \overline{\Delta^\alpha_\beta}$ are Maurer–Cartan forms for G modulo $\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \theta^n, \theta^{\bar{n}}$, and where $\hat{g}_{\bar{\alpha}\beta}$ and $\hat{h}^\alpha_{\bar{\beta}}$ are functions on $Y \cong M \times G$ satisfying the following

$$(2.19) \qquad u^{-1} \hat{g}_{\bar{\alpha}\beta}(p, ST) \overline{u^\alpha_\mu} u^\beta_\nu = \hat{g}_{\bar{\mu}\nu}(p, T), \quad \hat{h}^\alpha_{\bar{\beta}}(p, ST) \overline{u^\beta_\mu} = \hat{h}^\nu_{\bar{\mu}}(p, T) u^\alpha_\nu,$$

where $S, T \in G$ with S given by (2.6) and $p \in M$. The last action of G in (2.19) can also be described using the third order tensor $\hat{h}_{\bar{\alpha}\bar{\beta}n} = \hat{g}_{\bar{\alpha}\mu} h^\mu_{\bar{\alpha}}/2i$ by

$$(2.20) \qquad u^{-1} \hat{h}_{\bar{\alpha}\bar{\beta}n}(p, ST) \overline{u^\alpha_\mu} \overline{u^\beta_\nu} = \hat{h}_{\bar{\mu}\bar{\nu}n}(p, T).$$

In what follows, we shall simply denote $\hat{h}_{\bar{\alpha}\bar{\beta}n}$ by $\hat{h}_{\bar{\alpha}\bar{\beta}}$. We shall now proceed under two assumptions. The first is the following.

Condition 2.21. The matrix $(g_{\bar{\alpha}\beta}(p_0))$ is *definite*.

In view of the first identity in (2.19), we can make an initial choice of the θ, θ^α so that the matrix $(g_{\bar{\alpha}\beta})$ is constant with $g_{\bar{\alpha}\beta} = \delta_{\bar{\alpha}\beta}$. Denote by G' the subgroup of G consisting of those matrices S, as in (2.6), for which

$$(2.22) \qquad u^{-1} g_{\bar{\alpha}\beta} \overline{u^\alpha_\mu} u^\beta_\nu = g_{\bar{\mu}\nu},$$

i.e. those for which $u > 0$ and $(u^{-1/2} u^{\alpha}_\beta)$ is unitary. As a consequence of E. Cartan's work on Lie groups (see also [S] for an elementary proof of precisely the following statement), the (complex) symmetric matrix $(h_{\bar{\alpha}\bar{\beta}})$ can be conjugated by the action $v^\alpha_\beta \mapsto h_{\bar{\alpha}\bar{\beta}} \overline{v^\alpha_\mu} \overline{v^\beta_\nu}$, where (v^α_β) is unitary, to the form

$$(2.23) \qquad D_{n-1}(\lambda_{\bar{1}}, \ldots, \lambda_{\bar{n}-\bar{1}}) := \begin{pmatrix} \lambda_{\bar{1}} & 0 & \ldots & 0 \\ 0 & \lambda_{\bar{2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{\bar{n}-\bar{1}} \end{pmatrix},$$

where $\lambda_{\bar{1}} \geq \lambda_{\bar{2}} \geq \ldots \geq \lambda_{\bar{n}-\bar{1}} \geq 0$ are uniquely determined by $(h_{\bar{\alpha}\bar{\beta}})$. Moreover, the subgroup of the unitary matrices (v^{α}_β) for which

$$(2.24) \qquad \hat{g}_{\bar{\alpha}\beta} \overline{v^\alpha_\mu} v^\beta_\nu = \hat{g}_{\bar{\mu}\nu}, \quad \hat{h}_{\bar{\alpha}\bar{\beta}} \overline{v^\alpha_\mu} \overline{v^\beta_\nu} = \hat{h}_{\bar{\mu}\bar{\nu}}$$

can be described explicitly (see e.g. [E4, Lemma 5.24]). Our second assumption is the following.

Condition 2.25. The positive numbers $\lambda_{\bar{1}}(p_0), \ldots, \lambda_{\bar{n}-\bar{1}}(p_0)$ associated with the matrix $(h_{\bar{\alpha}\bar{\beta}}(p_0))$, as described above, are all distinct and nonzero, i.e.

$$(2.26) \qquad \lambda_{\bar{1}}(p_0) > \ldots > \lambda_{\bar{n}-\bar{1}}(p_0) > 0.$$

Under this assumption, there is a small neighborhood of p_0 for which $\lambda_{\bar{1}}(p) > \ldots > \lambda_{\bar{n}-\bar{1}}(p) > 0$ for p in that neighborhood. In view of the above and using the fact that $\hat{h}_{\bar{\alpha}\bar{\beta}}(p)$ can be changed by a scalar multiple by changing $\theta^n(p)$ by a scalar multiple, we see that is possible to choose the basis $\theta, \theta^\alpha, \theta^n$ so that $g_{\bar{\alpha}\beta}$ is constant, equal to $\delta_{\bar{\alpha}\beta}$, and $h_{\bar{\alpha}\bar{\beta}}$ satisfies

$$(2.27) \qquad \operatorname{tr} \hat{h}_{\bar{\alpha}\bar{\beta}}(p) = 1,$$

for p in a neighborhood of p_0 (which we from now on identify with M). Here, tr denotes the usual trace of a matrix; thus, we have $\operatorname{tr} \hat{h}_{\bar{\alpha}\bar{\beta}} = \lambda_{\bar{1}} + \ldots + \lambda_{\bar{n}-\bar{1}}$. It follows from [E4, Lemma 5.24] that the only unitary matrices $(v_\beta^\alpha) = (u^{-1/2} u_\beta^\alpha)$ now satisfying (2.24) are the diagonal matrices $D_{n-1}(\epsilon_1, \ldots, \epsilon_{n-1})$, where we have used the notation introduced in (2.23) and each $\epsilon_j \in \{-1, 1\}$. Let us denote by G_1' the subgroup of G' consisting of those S, as in (2.6), where $(u^{-1/2} u^{\alpha\beta})$ equals $D_{n-1}(\epsilon_1, \ldots, \epsilon_{n-1})$ with $\epsilon_j \in \{-1, 1\}$. Observe that the covectors $\omega^\alpha(p_0)$ define an ordered set of linear forms on $\bar{\mathcal{V}}_{p_0}$, each of which is invariantly defined up to multiplication by $\pm u^{-1/2}$ under the action of the group G_1' in (2.5). We shall refer to a choice of each of these linear forms (up to multiplication by a *positive* real number) as a choice of *orientation for the normalized CR structure* at p_0. Thus, given such a choice of orientation, the only matrices $S \in G_1'$ which preserve this orientation are the ones for which $(u^{-1/2} u^{\alpha\beta})$ equals the identity. In what follows, we shall assume that such a choice of orientation has been made. Let us also remark that if $n = 2$, then there is only one linear form $\omega^1(p_0)$ and its annihilator coincides with the Levi nullspace $\mathfrak{N}_{p_0}$. Hence, in this case, normalizing the second and third order tensor $\hat{g}_{\bar{\alpha}\beta}, \hat{h}_{\bar{\alpha}\bar{\beta}}$ as above determines, up to multiplication by a nonzero real number, a linear form defining the Levi nullspace $\mathfrak{N}_{p_0}$. A choice of orientation for the normalized CR structure at p_0 determines the sign of this real number. For this reason we shall refer to a choice of orientation for the normalized CR structure in the case $n = 2$ as a choice of *orientation for the Levi nullspace*.

We denote by G_1 the subgroup of $G_1' \subset G'$ consisting of those S, as in (2.6), where $(u^{-1/2} u^{\alpha\beta})$ equals the identity. It is easy to compute the Lie algebra $\mathfrak{g}_1$ of G_1. We have $T \in \mathfrak{g}_1$ if and only if

$$(2.28) \qquad T = \begin{pmatrix} 2v & 0 & 0 \\ v^\alpha & v\delta_\beta^\alpha & 0 \\ \overline{v^\alpha} & 0 & v\delta_\beta^\alpha \end{pmatrix}, \qquad (v, v^\alpha) \in \mathbb{R} \times \mathbb{C}^{n-1}.$$

Also, denote by $Y_1 \subset Y$ the principal bundle over M with group G_1 consisting of those $(\omega, \omega^\alpha, \omega^{\bar{\alpha}})^\tau$ for which $\hat{g}_{\bar{\alpha}\beta}$ and $\hat{h}_{\bar{\mu}}^\alpha$ in the structure equation (2.18) satisfy

$$(2.29) \qquad \hat{g}_{\bar{\alpha}\beta}(p, S) = \delta_{\bar{\alpha}\beta}, \quad \hat{h}_{\bar{\alpha}\bar{\beta}} = \lambda_{\bar{\beta}} \delta_{\bar{\alpha}\bar{\beta}},$$

i.e. $Y_1 \cong M \times G_1 \subset M \times G \cong Y$ where the trivializations are the ones obtained by

of Y_2 (cf. e.g. [S, Chapter VII]). In what follows, we shall, in order to make the formulas more invariant, continue to use the notation $g_{\bar{\alpha}\beta}$, $\hat{h}_{\bar{\alpha}\bar{\beta}}$ rather than the special form (2.29).

Taking the pullbacks of all the forms $\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \theta^n, \theta^{\bar{n}}, \Delta, \Delta^\alpha, \Delta^\alpha_\beta$ to Y_1, we see from formula (2.8) (with $S \in G_1$ now) that, pulled back to Y_1, the matrix

$$\underline{\Delta} = \begin{pmatrix} \Delta & 0 & 0 \\ \Delta^\alpha & \Delta^\alpha_\beta & 0 \\ \Delta^{\bar{\alpha}} & 0 & \Delta^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix}$$

is valued in $\mathfrak{g}_1$ modulo $(\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \theta^n, \theta^{\bar{n}})$, i.e.

$$(2.30) \qquad\qquad \Delta^\alpha_\beta = \Delta^{\bar{\alpha}}_{\bar{\beta}} = \frac{1}{2}\Delta\delta^\alpha_\beta \qquad \mod (\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \theta^n, \theta^{\bar{n}}).$$

Let us therefore rewrite (2.18) as follows

$$(2.31) \quad d\begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} \Delta & 0 & 0 \\ \Delta^\alpha & \frac{1}{2}\Delta\delta^\alpha_\beta & 0 \\ \Delta^{\bar{\alpha}} & 0 & \frac{1}{2}\Delta\delta^\alpha_\beta \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\beta \\ \omega^{\bar{\beta}} \end{pmatrix} + \begin{pmatrix} i\hat{g}_{\bar{\mu}\nu}\omega^{\bar{\mu}} \wedge \omega^\nu \\ \hat{t}^\alpha_{\bar{\mu}\nu}\omega^{\bar{\mu}} \wedge \omega^\nu \\ \hat{t}^\alpha_{\bar{\mu}\nu}\omega^\mu \wedge \omega^{\bar{\nu}} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \hat{h}^\alpha_{\bar{\mu}}\omega^{\bar{\mu}} \wedge \theta^n + \hat{q}^\alpha_{\nu\beta}\omega^\beta \wedge \omega^\nu + \hat{r}^\alpha_\nu\omega^\nu \wedge \theta^n + \hat{s}^\alpha_\nu\theta^{\bar{n}} \wedge \omega^\nu \\ \hat{h}^\alpha_{\bar{\mu}}\omega^\mu \wedge \theta^{\bar{n}} + \overline{\hat{q}^\alpha_{\nu\beta}}\omega^{\bar{\beta}} \wedge \omega^{\bar{\nu}} + \overline{\hat{r}^\alpha_\nu}\omega^{\bar{\nu}} \wedge \theta^{\bar{n}} + \overline{\hat{s}^\alpha_\nu}\theta^n \wedge \omega^{\bar{\nu}} \end{pmatrix},$$

where, by a slight abuse of notation, all the forms in (2.31) denote the pulled back forms on Y_1; we require the $\hat{q}^\alpha_{\nu\beta}$ to be skew symmetric in ν, β. The 1-form Δ is uniquely determined by the form of the structure equation (2.31) up to transformations

$$(2.32) \qquad\qquad\qquad \tilde{\Delta} = \Delta + a\omega,$$

where a is a smooth function on Y_1; i.e. replacing Δ by $\tilde{\Delta}$ given by (2.32) preserves (2.31) and this is the only transformation preserving (2.31), as is easily verified (cf. also [G, Lecture 3]). In fact, a direct calculation shows that

$$\Delta = u^{-1}du + i\hat{g}_{\bar{\mu}\nu}u^{-1/2}(\overline{u^\mu}\omega^\nu - u^\nu\omega^{\bar{\mu}}) + \phi \mod \omega.$$

where ϕ is the pullback of a real 1-form on M. The 1-form θ^n on Y_1 is determined by the condition tr $\hat{h}_{\bar{\alpha}\bar{\beta}} \equiv 1$ up to transformations

$$(2.33) \qquad\qquad\qquad \tilde{\theta}^n = \theta^n + c_\beta\omega^\beta + c\omega,$$

where c, c_β are smooth functions on Y_1. The 1-forms Δ^α are determined modulo $\omega, \omega^\nu, \omega^{\bar{\mu}}, \theta^n, \theta^{\bar{n}}$. The precise form of the indeterminacy in Δ^ν is not important at this point.

By using the integrability of $\mathcal{V}$ and the fact that tr $\hat{h}_{\bar{\alpha}\bar{\beta}}$ is constant on the fibers $Y_1 \to M$, we deduce that we can write

$$d\tilde{\theta}^n = d\theta^n + dc_\beta \wedge \omega^\beta + c_\beta d\omega^\beta + dc \wedge \omega + cd\omega$$

$$(2.34)$$

where η and η_β are 1-forms (depending also on c_β and c),

$$(2.35) \qquad \tilde{k}_{\bar{\mu}} = \hat{k}_{\bar{\mu}} + c_\nu \hat{h}_{\bar{\mu}}^\nu - \overline{c_\mu}\hat{k},$$

and $\hat{k}_{\bar{\mu}}$, $\hat{k}$ are smooth functions on Y_1. Moreover, $\hat{k}$ is constant on the fibers $Y_1 \to M$, whereas $\hat{k}_\mu(p, ST) = u^{-1/2}\hat{k}_{\bar{\mu}}(p, T)$ for $S, T \in G_1$ and S of the form (2.6). The reader can easily verify that the function $\hat{k}$ is also uniquely determined, i.e. independent of the choice of $\Delta, \Delta^\nu, \theta^n$. It is also easy to check that the functions $\hat{r}_\nu^\alpha$, $\hat{s}_\nu^\alpha$ in (2.31) are uniquely determined, and that they are constant on the fibers $Y_1 \to M$. Hence, $\hat{k}(p)$, $\hat{r}_\nu^\alpha(p)$, $\hat{s}_\nu^\alpha(p)$ are invariants of the CR structure on M at $p \in M$. However, they are not independent as the following proposition shows.

Proposition 2.36. *We have the following identities*

$$(2.37) \qquad \hat{g}_{\bar{\mu}\eta}\hat{s}_\nu^\eta = \hat{g}_{\bar{\xi}\nu}\overline{\hat{r}_\mu^\xi} = \overline{\hat{g}_{\bar{\nu}\xi}\hat{r}_\mu^\xi},$$

and

$$(2.38) \qquad \hat{k} = \text{tr} \left(\hat{h}_{\bar{\alpha}\mu}\overline{\hat{r}_\beta^\mu} + \hat{h}_{\bar{\mu}\bar{\beta}}\overline{\hat{r}_\alpha^\mu} \right)_{\alpha\beta}.$$

Proof. Observe that the second identity in (2.37) follows directly from the fact that $(\hat{g}_{\bar{\alpha}\beta})$ is Hermitian. Differentiating the first row in the structure equation (2.31), using the facts that $d^2\omega = 0$ and $\hat{g}_{\bar{\alpha}\beta}$ is constant on Y_1, we obtain

$$(2.39) \qquad 0 = d\Delta \wedge \omega - \Delta \wedge d\omega + i\hat{g}_{\bar{\mu}\nu}(d\omega^{\bar{\mu}} \wedge \omega^\nu - \omega^{\bar{\mu}} \wedge d\omega^\nu).$$

Applying equation (2.31) again, we obtain

$$(2.40) \quad 0 = \left(d\Delta + i\hat{g}_{\bar{\mu}\nu}(\Delta^\nu \wedge \omega^{\bar{\mu}} - \Delta^{\bar{\mu}} \wedge \omega^\nu)\right) \wedge \omega$$

$$+ i\hat{g}_{\bar{\mu}\nu}(\overline{\hat{t}_{\bar{\xi}\eta}^\mu}\omega^\xi \wedge \omega^{\bar{\eta}} \wedge \omega^\nu - \hat{t}_{\bar{\xi}\eta}^\nu\omega^{\bar{\xi}} \wedge \omega^\eta \wedge \omega^{\bar{\mu}}) + i\left(\hat{g}_{\bar{\xi}\nu}\overline{\hat{s}_\mu^\xi} - \hat{g}_{\bar{\mu}\eta}\hat{r}_\nu^\eta\right)\theta^n \wedge \omega^\mu \wedge \omega^\nu$$

$$+ i\left(\hat{g}_{\bar{\mu}\eta}\hat{s}_\nu^\eta - \hat{g}_{\bar{\xi}\nu}\overline{\hat{r}_\mu^\xi}\right)\theta^{\bar{n}} \wedge \omega^{\bar{\mu}} \wedge \omega^\nu - i\hat{g}_{\bar{\mu}\nu}(\overline{\hat{h}_{\bar{\xi}}^\mu}\theta^{\bar{n}} \wedge \omega^\xi \wedge \omega^\nu + \hat{h}_{\bar{\eta}}^\nu\theta^n \wedge \omega^{\bar{\mu}} \wedge \omega^{\bar{\eta}}).$$

The first identity in (2.37) follows immediately from (2.40). To prove (2.38), we differentiate the formula for $d\omega^\alpha$ given by (2.31). We obtain

$$(2.41) \quad 0 = d\Delta^\alpha \wedge \omega - \Delta^\alpha \wedge d\omega + \frac{1}{2}(d\Delta \wedge \omega^\alpha - \Delta \wedge d\omega^\alpha) + d\hat{t}_{\bar{\mu}\nu}^\alpha \wedge \omega^{\bar{\mu}} \wedge \omega^\nu +$$

$$\hat{t}_{\bar{\mu}\nu}^\alpha(d\omega^{\bar{\mu}} \wedge \omega^\nu - \omega^{\bar{\mu}} \wedge d\omega^\nu) + d\hat{h}_{\bar{\mu}}^\alpha \wedge \omega^{\bar{\mu}} \wedge \theta^n + \hat{h}_{\bar{\mu}}^\alpha(d\omega^{\bar{\mu}} \wedge \theta^n - \omega^{\bar{\mu}} \wedge d\theta^n) +$$

$$d\hat{r}_\nu^\alpha \wedge \omega^\nu \wedge \theta^n + \hat{r}_\nu^\alpha(d\omega^\nu \wedge \theta^n - \omega^\nu \wedge d\theta^n) + d\hat{s}_\nu^\alpha \wedge \theta^{\bar{n}} \wedge \omega^\nu + \hat{s}_\nu^\alpha(d\theta^{\bar{n}} \wedge \omega^\nu - \theta^{\bar{n}} \wedge d\omega^\nu).$$

Let us write $d\hat{h}_{\bar{\mu}}^\alpha = e_{\bar{\mu}}^\alpha\theta^{\bar{n}}$ modulo $\omega, \omega^\nu, \omega^{\bar{\nu}}, \theta^n$. If we substitute (2.31) and (2.35) in (2.41) and collect the $\omega^{\bar{\mu}} \wedge \theta^{\bar{n}} \wedge \theta^n$-terms, we obtain

$$(2.42) \qquad \hat{k}\hat{h}_{\bar{\mu}}^\alpha = \hat{h}_{\bar{\gamma}}^\alpha\overline{\hat{r}_\mu^\gamma} + \hat{s}_\gamma^\alpha\hat{h}_{\bar{\mu}}^\gamma - e_{\bar{\mu}}^\alpha.$$

Multiplying (2.42) by $2i\hat{g}_{\bar{\beta}\alpha}$, summing over α, and using (2.37), we obtain

$$(2.43) \qquad \hat{k}\hat{h}_{\bar{\mu}} = \hat{h}_{\bar{\gamma}}\overline{\hat{s}^\gamma} + \hat{h} \qquad \overline{\hat{s}^\gamma} \qquad 2i\hat{g} \qquad c^\alpha$$

The identity (2.38) now follows by taking the trace of (2.43) and using the fact that tr $\hat{h}_{\alpha\bar{\beta}} \equiv 1$. (In particular, tr $\hat{g}_{\bar{\beta}\alpha}e^\alpha_{\bar{\mu}} \equiv 0$.) □

For future reference, let us remark that the functions $\hat{t}^\alpha_{\bar{\mu}\nu}$ in (2.31) depend on the choice of θ^n. For a fixed such choice, the functions $\hat{t}^\alpha_{\bar{\mu}\nu}$ satisfy the following on Y_1

$$(2.44) \qquad \hat{t}^\alpha_{\bar{\mu}\nu}(p, ST) = \frac{\hat{t}^\alpha_{\bar{\mu}\nu}(p, T)}{\sqrt{u}} + i\hat{g}_{\bar{\mu}\nu}(p)\frac{u^\alpha}{u} + i\frac{1}{2}\hat{g}_{\bar{\mu}\gamma}(p)\frac{u^\gamma}{u}\delta^\alpha_\nu,$$

where $S, T \in G_1$ and S is given by (2.6). Let us also observe that replacing θ^n by $\tilde{\theta}^n$, as given by (2.33), the $\hat{t}^\alpha_{\bar{\mu}\nu}$ change by

$$(2.45) \qquad \hat{t}^\alpha_{\bar{\mu}\nu} \to \hat{t}^\alpha_{\bar{\mu}\nu} - (\hat{h}^\alpha_{\bar{\mu}}c_\nu + \hat{s}^\alpha_\nu\overline{c_\mu}).$$

3. The 5 dimensional case

We now restrict our attention to the case $n = 2$. The conditions 2.21 and 2.25 reduce to requiring that M is Levi uniform of rank one and 2-nondegenerate at p_0. Moreover, we have fixed an orientation of the Levi nullspace $\mathfrak{N}_{p_0}$, as explained in section 2, in order to distinguish a component Y_1 of the disconnected bundle Y_1'. Our aim is to define a uniquely determined submanifold $Y_2 \subset Y_1$ which can be viewed as a principal fiber bundle over M whose group is a subgroup of G_1, and on the principal bundle $Y_2 \to M$ determine the forms θ^n, Δ, and Δ^1 uniquely. (Recall that $n = 2$ so that Greek indices α, β run over the single integer 1.) We have to distinguish two cases. We shall begin by handling the most difficult case, which does not appear to be the generic one but which contains the tube over the light cone as given by Example 1.7. Observe in what follows that the condition tr $\hat{h}_{\alpha\bar{\beta}} \equiv 1$ reduces to $\hat{h}_{\bar{1}1} \equiv 1$. To simplify the notation, we shall use the fact that $\hat{g}_{\bar{1}1} \equiv 1$. Also, recall the invariant $\hat{k}$ defined by (2.34).

3.1. Case 1: $|\hat{k}(p_0)| = 2$. Consider the formula in (2.35) for the coefficient $\tilde{k}_{\bar{1}}$ of $\omega^{\bar{1}} \wedge \tilde{\theta}^2$ in $d\tilde{\theta}^2$ and rewrite it in the following form

$$(3.1.1) \qquad \tilde{k}_{\bar{1}} = \hat{k}_{\bar{1}} + 2ic_1 - \overline{c_1}\hat{k}.$$

Since $|\hat{k}(p_0)| = 2$, it is not possible to solve the equation $\tilde{k}_{\bar{1}} = 0$ for c_1 in a neighborhood of p_0. Let us write

$$(3.1.2) \qquad \hat{k} = 2ire^{it},$$

where r and t are real valued functions on Y_1 which are constant on the fibers of $Y_1 \to M$. We shall seek c_1 in the form

$$(3.1.3) \qquad c_1 = e^{it/2}(\rho_1 + i\rho_2),$$

where $\rho_1, \rho_2 \in \mathbb{R}$. Since $r(p_0) = 1$ we can choose ρ_2 uniquely so that

$$\tilde{k} \qquad \qquad e^{it/2}$$

for some real valued function r' on Y_1 which is constant on the fibers of $Y_1 \to M$. The function ρ_2 satisfies the same transformation rule as $\hat{k}_{\bar{1}}$, i.e.

$$(3.1.4) \qquad\qquad \rho_2(p, ST) = u^{-1/2} \rho_2(p, T),$$

for $S, T \in G_1$ and S of the form (2.6). We can express the above by saying that $\mathrm{Im}\, e^{-it/2} c_1$ is uniquely determined by the condition

$$(3.1.5) \qquad\qquad \mathrm{Re}\, e^{-it/2} \hat{k}_{\bar{1}} \equiv 0.$$

Now, with $\mathrm{Im}\, e^{-it/2} c_1$ determined by (3.1.5), the equation (2.45) implies that the function $\hat{t}^1_{\bar{1}1}$, given by (2.31), is determined up to

$$\hat{t}^1_{\bar{1}1} \to \hat{t}^1_{\bar{1}1} - \left(2i\rho_1 e^{it/2} + \hat{s}^1_1 \rho_1 e^{-it/2} \right)$$

or equivalently, in view of Proposition 2.36 and (3.1.2),

$$
\begin{aligned}
(3.1.6) \qquad
\hat{t}^1_{\bar{1}1} \to & \hat{t}^1_{\bar{1}1} - 2i \left(\rho_1 e^{it/2} + \frac{1}{2} \hat{k} \rho_1 e^{-it/2} \right) \\
= & \hat{t}^1_{\bar{1}1} - 2i\rho_1 e^{it/2} \left(1 + \frac{1}{2} r \right).
\end{aligned}
$$

Let us write $u^1 = e^{it/2}(x + iy)$. It follows from (2.44) that there is a submanifold $Y_2 \subset Y_1$, which is defined (uniquely in view of (3.1.6)) by the equation

$$(3.1.7) \qquad\qquad \mathrm{Re}\, (e^{-it/2} \hat{t}^1_{\bar{1}1}) = 0.$$

The manifold Y_2 can be viewed as a principal fiber bundle $Y_2 \to M$ with group $G_2 \subset G_1$, where G_2 is defined by the equation

$$(3.1.8) \qquad\qquad \mathrm{Im}\, e^{-it(p_0)/2} u^1 = 0,$$

if we let G_2 act on Y_2 as follows

$$(3.1.9) \qquad g \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} := \begin{pmatrix} u & 0 & 0 \\ e^{it/2}x & \sqrt{u} & 0 \\ e^{-it/2}x & 0 & \sqrt{u} \end{pmatrix} \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix},$$

for $g \in G_2 \subset G_1$ of the form

$$(3.1.10) \qquad\qquad \begin{pmatrix} u & 0 & 0 \\ e^{it(p_0)/2}x & \sqrt{u} & 0 \\ e^{-it(p_0)/2}x & 0 & \sqrt{u} \end{pmatrix}.$$

Observe that the principal G_2 bundle Y_2 is a reduction of the principal G_1 bundle Y_1 only if the phase function t is constant. By definition, we have $\hat{t}^1_{\bar{1}1} = ir'' e^{it/2}$, for some real valued function r'', on the bundle Y_2. Hence, we can determine ρ_1 uniquely in (3.1.6) so that

(3.1.11) ⌢̂¹ = 0

on Y_2. It follows from (2.44) and (3.1.6) that we have

$$(3.1.12) \qquad \rho_1(p, ST) = \frac{\rho_1(p, T)}{\sqrt{u}} + \frac{3}{2(2+r)} \frac{x}{u},$$

for $S, T \in G_2$ and S of the form (2.6) with $u^1 = e^{it/2}x$ and $x \in \mathbb{R}$.

Thus, θ^2 is determined on Y_2 up to

$$(3.1.13) \qquad \tilde{\theta}^2 = \theta^2 + c\omega$$

by (3.1.5) and (3.1.11). Observe that on Y_2, where $u^1 = e^{it/2}x$ for $x \in \mathbb{R}$, we have

$$(3.1.14) \qquad \Delta^1 = e^{it/2}\xi \qquad \mod \omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}},$$

where ξ is a real 1-form on Y_2 which is not uniquely determined. We can write the structure equation for $d\omega^1$ on Y_2 as follows

$$(3.1.15) \quad d\omega^1 = e^{it/2}\xi \wedge \omega + \frac{1}{2}\Delta \wedge \omega^1 + \hat{h}_1^1 \omega^{\bar{1}} \wedge \theta^2 + \hat{r}_1^1 \omega^1 \wedge \theta^2 + \hat{s}_1^1 \theta^{\bar{2}} \wedge \omega^1$$
$$+ ie^{it/2}(b\omega^1 \wedge \omega + \bar{b}\omega^{\bar{1}} \wedge \omega + e\theta^2 \wedge \omega + \bar{e}\theta^{\bar{2}} \wedge \omega),$$

for some functions b and e on Y_2. Using (2.32) and (3.1.13), we obtain

$$(3.1.16) \quad d\omega^1 = e^{it/2}\tilde{\xi} \wedge \omega + \frac{1}{2}\tilde{\Delta} \wedge \omega^1 + \hat{h}_1^1 \omega^{\bar{1}} \wedge \tilde{\theta}^2 + \hat{r}_1^1 \omega^1 \wedge \tilde{\theta}^2 + \hat{s}_1^1 \tilde{\theta}^{\bar{2}} \wedge \omega^1$$
$$+ ie^{it/2}(\tilde{b}\omega^1 \wedge \omega + \overline{\tilde{b}}\omega^{\bar{1}} \wedge \omega + e\tilde{\theta}^2 \wedge \omega + \bar{e}\tilde{\theta}^{\bar{2}} \wedge \omega),$$

where

$$(3.1.17) \qquad \tilde{b} = b + \frac{1}{2}\left(\frac{1}{2}ae^{-it/2} + \bar{c}\hat{s}_1^1 e^{-it/2} + \bar{c}\overline{\hat{h}_1^1}e^{it/2} - c\hat{r}_1^1 e^{-it/2}\right)$$

and

$$(3.1.18) \quad \tilde{\xi} = \xi + \frac{1}{2}\left(\frac{1}{2}(ae^{-it/2}\omega^1 + \bar{a}e^{it/2}\omega^{\bar{1}}) + \bar{c}\hat{s}_1^1 e^{-it/2}\omega^1 + c\overline{\hat{s}_1^1}e^{it/2}\omega^{\bar{1}}\right)$$
$$- \frac{1}{2}\left(\bar{c}\overline{\hat{h}_1^1}e^{it/2}\omega^1 + c\hat{h}_1^1 e^{-it/2}\omega^{\bar{1}} + c\hat{r}_1^1 e^{-it/2}\omega^1 + \bar{c}\overline{\hat{r}_1^1}e^{it/2}\omega^{\bar{1}}\right) + q\omega.$$

Here, q is an arbitrary real valued function on Y_2, and a and c are as in (2.32) and (3.1.13) respectively. We deduce that ξ is determined by the structure equation (3.1.16) up to transformations given by (3.1.18). Let us rewrite (3.1.17) and (3.1.18) using Proposition 2.36 and (3.1.2) as follows

$$(3.1.19) \qquad \tilde{b} = b + \frac{1}{2}e^{-it/2}\left(\frac{1}{2}a + i(r-2)\bar{\zeta} + ir\zeta\right),$$

and

$$(3.1.20) \quad \tilde{\xi} = \xi + \frac{1}{2}e^{-it/2}\left(\frac{1}{2}a + i(r+2)\bar{\zeta} + ir\zeta\right)\omega^1$$
$$+ \frac{1}{2}e^{it/2}\left(\frac{1}{2}\bar{a} - i(r+2)\zeta - ir\bar{\zeta}\right)\omega^{\bar{1}} + q\omega,$$

where we have used the notation

$$\zeta = e^{-it}c.$$

By (2.40), we have on Y_1,

(3.1.21)
$$d\Delta = i\hat{g}_{\bar{1}1}(\Delta^{\bar{1}} \wedge \omega^1 - \Delta^1 \wedge \omega^{\bar{1}}) + \Phi \wedge \omega$$
$$= i(\Delta^{\bar{1}} \wedge \omega^1 - \Delta^1 \wedge \omega^{\bar{1}}) + \Phi \wedge \omega,$$

for some real 1-form Φ; on the second line of (3.1.21), we have used $\hat{g}_{\bar{1}1} = 1$. Hence, on Y_2 we have, by (3.1.14),

(3.1.22) $\qquad d\Delta = ie^{-it/2}\xi \wedge \omega^1 - ie^{it/2}\xi \wedge \omega^{\bar{1}} + \Psi \wedge \omega + \Psi_1 \wedge \omega^1 + \Psi_{\bar{1}} \wedge \omega^{\bar{1}},$

for some 1-forms $\Psi, \Psi_1, \Psi_{\bar{1}} = \overline{\Psi_1}$ on Y_2 such that $\Psi_1 = 0$ modulo $\omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}$. Recall that Δ is determined up to transformations (2.32), θ^2 up to transformations (3.1.13), and ξ up to transformations (3.1.20). Substituting in (3.1.22), we obtain

(3.1.23)
$$d\tilde{\Delta} = d\Delta + da \wedge \omega + ad\omega$$
$$= ie^{-it/2}\tilde{\xi} \wedge \omega^1 - ie^{it/2}\tilde{\xi} \wedge \omega^{\bar{1}} + i\tilde{f}\omega^{\bar{1}} \wedge \omega^1 + \dots,$$

where $\dots$ signify the remaining terms in the expansion of $d\tilde{\Delta}$ and

(3.1.24)
$$\tilde{f} = f - \frac{1}{2}a + i\zeta - i\bar{\zeta},$$

for some real valued function f on Y_2. Hence, a is uniquely determined as a function of ζ by the condition

(3.1.25)
$$\tilde{f} = 0,$$

and we have

(3.1.26)
$$a = 2i(\zeta - \bar{\zeta}).$$

In view of (3.1.3), (3.1.4), and (3.1.12), we have

(3.1.27)
$$dc_1 = e^{it/2}\frac{3}{2(2+r)}\xi \quad \mathrm{mod}\ \Delta, \omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}.$$

It follows, by using (2.34), (3.1.20) and (3.1.24), that

(3.1.28)
$$d\tilde{\theta}^2 = e^{it/2}\frac{3}{2(2+r)}\tilde{\xi} \wedge \omega^1 + \tilde{m}\omega^{\bar{1}} \wedge \omega^1 + \dots,$$

where $\dots$ signify the remaining terms in the expansion of $d\tilde{\theta}^2$ and $\tilde{m}$ is given by

(3.1.29)
$$\tilde{m} = m + \frac{3ie^{it}}{4}\frac{(r+1)}{(r+2)}\left(\zeta + \bar{\zeta}\right) + ic$$
$$= m + \frac{ie^{it}}{4(r+2)}\left((7r+11)\zeta + 3(r+1)\bar{\zeta}\right)$$

for some function m on Y_2; in the last line of (3.1.29), we have used $c = e^{it}\zeta$. Since $r(p_0) = 1$, we can determine ζ uniquely by the condition

$$(3.1.30) \qquad\qquad\qquad \tilde{m} = 0.$$

Let us summarize our efforts so far. We have determined Δ and θ^2 uniquely on Y_2. The 1-form ξ is determined up to

$$(3.1.31) \qquad\qquad\qquad \tilde{\xi} = \xi + q\omega,$$

for some real valued function q on Y_2. We shall conclude the construction in this section by defining a unique choice of ξ. In view of (3.1.22), we have

$$(3.1.32) \qquad d\Delta = ie^{-it/2}\tilde{\xi} \wedge \omega^1 - ie^{it/2}\tilde{\xi} \wedge \omega^{\bar{1}} + \tilde{l}\omega^1 \wedge \omega + \overline{\tilde{l}}\omega^{\bar{1}} \wedge \omega + \dots,$$

where

$$(3.1.33) \qquad\qquad\qquad \tilde{l} = l + ie^{-it/2}q$$

for some function l on Y_2. Hence, we may determine q uniquely by the condition

$$(3.1.34) \qquad\qquad\qquad \operatorname{Im} e^{it/2}\tilde{l} = 0.$$

The uniquely determined, linearly independent 1-forms

$$(3.1.35) \qquad\qquad \omega, \operatorname{Re} \omega^1, \operatorname{Im} \omega^1, \operatorname{Re} \theta^2, \operatorname{Im} \theta^2, \Delta, \xi$$

on Y_2 form a global coframe for T^*Y_2 and, hence, define an absolute parallelism on Y_2. The following result is a consequence of the construction above (see [G] or [CM]). We use the notation introduced previously, and also

$$(3.1.36) \qquad\qquad\qquad \underline{\omega} := \begin{pmatrix} \omega \\ \operatorname{Re} \omega^1 \\ \operatorname{Im} \omega^1 \\ \operatorname{Re} \theta^2 \\ \operatorname{Im} \theta^2 \\ \Delta \\ \xi \end{pmatrix}.$$

Theorem 3.1.37. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Suppose that $|\hat{k}(p_0)| = 2$ and that an orientation is chosen for the Levi nullspace $\mathfrak{N}_{p_0}$ (as explained in §2). Then, there exists a principal fiber bundle $Y_2 \to M$ with a two dimensional structure group $G_2 \subset GL(\mathbb{C}^3)$ and a 1-form $\underline{\omega}$ on Y_2 which defines an isomorphism between $T_y Y_2$ and $\mathbb{R}^7$ for every $y \in Y_2$ and such that the following holds. Let M' be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0' \in M$. Suppose that the invariant $|\hat{k}'(p_0)| = 2$ (where corresponding objects for M' are denoted with $'$) and that an orientation is chosen for the Levi nullspace $\mathfrak{N}'_{p_0'}$. Then, if there exists a local CR*

and $\mathfrak{N}'_{p'_0}$, *there exists a diffeomorphism* $F \colon Y_2 \to Y'_2$ *with* $\pi' \circ F(\pi^{-1}(p_0)) = \{p'_0\}$ *such that* $F^*\underline{\omega}' = \underline{\omega}$ *and the following diagram commutes*

(3.1.38)

$$
\begin{array}{ccc}
Y_2 & \xrightarrow{\ F\ } & Y'_2 \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\
M & \xrightarrow[\ f\ ]{} & M'.
\end{array}
$$

Conversely, if there exists a diffeomorphism $F \colon Y_2 \to Y'_2$ *with* $\pi' \circ F(\pi^{-1}(p_0)) = p'_0$ *such that* $F^*\underline{\omega}' = \underline{\omega}$*, then there exists a CR diffeomorphism* $f \colon (M, p_0) \to (M', p'_0)$ *preserving the oriented Levi nullspaces* $\mathfrak{N}_{p_0}$ *and* $\mathfrak{N}'_{p'_0}$*, such that* (3.1.38) *commutes.*

   The 1-form $\underline{\omega}$ *is given by* (3.1.36) *and is uniquely determined by* (3.1.5)*,* (3.1.11)*,* (3.1.25)*,* (3.1.30)*, and* (3.1.34)*. The group* G_2 *is defined by* (3.1.10)*.*

3.2. Case 2: $|\hat{k}(p_0)| \neq 2$. First, since $|\hat{k}(p_0)| \neq 2$, there is a small neighborhood of p_0 in M in which $|\hat{k}| \neq 2$. In this neighborhood, which we identify with M in this subsection, we can solve uniquely for c_1 in the equation

(3.2.1)
$$
\tilde{k}_{\bar{1}} = 0,
$$

where $\tilde{k}_{\bar{1}}$ is given by equation (3.1.1). Thus, c_1 is uniquely determined by the condition (3.2.1), and hence θ^2 is determined up to transformations of the form (3.1.13) . Moreover, it follows from (2.44) that there exists a uniquely determined submanifold $Y_3 \subset Y_1$ defined by

(3.2.2)
$$
\hat{t}^1_{\bar{1}1} = 0.
$$

The submanifold Y_3 is a subbundle (a reduction) of the principal G_1-bundle Y_1 with group G_3, where G_3 is the subgroup of G_1 defined by

(3.2.3)
$$
u^1 = 0.
$$

Thus, on Y_3 we have $\Delta^1 = 0$ modulo $\omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}$. It follows that the structure equation for $d\omega^1$ on Y_3 can be written

$$
\text{(3.2.4)} \quad d\omega^1 = \frac{1}{2}\tilde{\Delta} \wedge \omega^1 + \hat{h}^1_1 \omega^{\bar{1}} \wedge \tilde{\theta}^2 + \hat{r}^1_1 \omega^1 \wedge \tilde{\theta}^2 + \hat{s}^1_1 \tilde{\theta}^{\bar{2}} \wedge \omega^1
$$
$$
+ {}^1\tilde{b}\omega^1 \wedge \omega + {}^2 b \tilde{\omega}^{\bar{1}} \wedge \omega + {}^1 e\theta^2 \wedge \omega + {}^2 e\theta^{\bar{2}} \wedge \omega,
$$

where

(3.2.5)
$$
{}^1\tilde{b} := {}^1 b - \hat{r}^1_1 c + \hat{s}^1_1 \bar{c} + \frac{1}{2}a, \quad {}^2\tilde{b} := {}^2 b - \hat{h}^1_1 c
$$

for some functions ${}^1 b$, ${}^2 b$, ${}^1 e$, and ${}^2 e$ on Y_3. We can determine c, and hence θ^2, uniquely on Y_3 by the condition

(3.2.6) ${}^2\tilde{b} = 0$

We then determine a, and hence Δ, uniquely on Y_3 by the condition

$$(3.2.7) \qquad\qquad \operatorname{Re} {}^1\tilde{b} = 0.$$

The uniquely determined 1-forms

$$\omega, \operatorname{Re} \omega^1, \operatorname{Im} \omega^1, \operatorname{Re} \theta^2, \operatorname{Im} \theta^2,$$

on Y_3 form a global coframe for T^*Y_3 and, hence, define an absolute parallelism on Y_3. As in 3.1, we have the following result. We use the notation introduced above, and also

$$(3.2.8) \qquad\qquad \underline{\omega} := \begin{pmatrix} \omega \\ \operatorname{Re} \omega^1 \\ \operatorname{Im} \omega^1 \\ \operatorname{Re} \theta^2 \\ \operatorname{Im} \theta^2 \\ \Delta \end{pmatrix}$$

Theorem 3.2.9. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Suppose that $|\hat{k}(p_0)| \neq 2$ and that an orientation is chosen for the Levi nullspace $\mathfrak{N}_{p_0}$ (as explained in §2). Then, there exists a principal fiber bundle $Y_3 \to M$ with a one dimensional structure group $G_3 \subset GL(\mathbb{C}^3)$ and a 1-form $\underline{\omega}$ on Y_3 which defines an isomorphism between $T_y Y_3$ and $\mathbb{R}^6$ for every $y \in Y_3$ and such that the following holds. Let M' be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0' \in M$. Suppose that the invariant $|\hat{k}'(p_0)| \neq 2$ (where corresponding objects for M' are denoted with $'$) and that an orientation is chosen for the Levi nullspace $\mathfrak{N}'_{p_0'}$. Then, if there exists a local CR diffeomorphism $f\colon (M, p_0) \to (M', p_0')$ preserving the oriented Levi nullspaces $\mathfrak{N}_{p_0}$ and $\mathfrak{N}'_{p_0'}$, there exists a diffeomorphism $F\colon Y_3 \to Y_3'$ with $\pi' \circ F(\pi^{-1}(p_0)) = \{p_0'\}$ such that $F^*\underline{\omega}' = \underline{\omega}$ and the following diagram commutes*

$$(3.2.10) \qquad \begin{array}{ccc} Y_3 & \xrightarrow{\;F\;} & Y_3' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow[\;f\;]{} & M'. \end{array}$$

Conversely, if there exists a diffeomorphism $F\colon Y_3 \to Y_3'$ with $\pi' \circ F(\pi^{-1}(p_0)) = p_0'$ such that $F^*\underline{\omega}' = \underline{\omega}$, then there exists a CR diffeomorphism $f\colon (M, p_0) \to (M', p_0')$ preserving the oriented Levi nullspaces $\mathfrak{N}_{p_0}$ and $\mathfrak{N}'_{p_0'}$, such that (3.2.10) commutes.

The 1-form $\underline{\omega}$ is given by (3.2.8) and is uniquely determined by (3.2.1), (3.2.6), and (3.2.7). The group G_3 is defined by (3.2.3), and on Y_3 (3.2.2) holds.

4. A CURVATURE CHARACTERIZATION OF THE TUBE OVER THE LIGHT CONE IN $\mathbb{C}^3$

In this section, we shall continue to consider only the case $n = 2$. We shall

A, B, etc. run over the set $\{1, 2\}$ (i.e. $\{1, \ldots, n\}$ with $n = 2$) and instead let them run over the set $\{1, 2, 3\}$. We shall also use the convention that small Roman indices a, b, etc. run over the set $\{0, 1, 2, 3\}$.

Denote by Γ the light cone in $\mathbb{R}^3$, i.e. the zero locus of the quadratic form $\{x, x\}$ where $\{\cdot, \cdot\}$ denotes the bilinear form which in the standard coordinates of $\mathbb{R}^3$ is given by

$$\{x, y\} := x^1 y^1 + x^2 y^2 - x^3 y^3. \tag{4.1}$$

We shall denote by $\Gamma_{\mathbb{C}}$ the tube in $\mathbb{C}^3$ over Γ (as in Example 1.7). Hence, $\Gamma_{\mathbb{C}}$ is the zero locus of $\{Z, Z\}_{\mathbb{C}}$, where $\{z, w\}_{\mathbb{C}}$ for complex vectors $z, w \in \mathbb{C}^3$ is defined by

$$\{z, w\}_{\mathbb{C}} := \{\text{Re } z, \text{Re } w\}. \tag{4.2}$$

We shall call a *frame* for $\Gamma_{\mathbb{C}}$ a 4-tuple (Z_0, Z_1, Z_2, Z_3) of 4-vectors, where

$$Z_0 = t(1, z_0), \quad Z_A = (0, x_A), \quad t \in \mathbb{R}, \ z_0 \in \mathbb{C}^3, \ x_A \in \mathbb{R}^3, \tag{4.3}$$

which satisfy the following conditions. The real vectors $x_1, x_2, x_3 \in \mathbb{R}^3$ satisfy

$$\begin{aligned} \{x_1, x_1\} = \{x_3, x_3\} = \{x_1, x_2\} = \{x_2, x_3\} = 0, \\ \{x_1, x_3\} = -1, \ \{x_2, x_2\} = 1 \end{aligned} \tag{4.4}$$

and also,

$$\text{Re } t z_0 = x_1. \tag{4.5}$$

Observe that the conditions (4.4) are equivalent to the fact that the (symmetric) matrix representation of the bilinear form $\{\cdot, \cdot\}$ relative to the basis x_1, x_2, x_3 is by the matrix Λ, where

$$\Lambda := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

We also write

$$\Lambda = (\lambda_{AB}). \tag{4.7}$$

The set of all frames for $\Gamma_{\mathbb{C}}$ can be viewed as a real subgroup of the complex Lie group $GL(\mathbb{C}^4)$ as follows. If (Z_0, Z_A) is a given frame, then any other frame (Z_0', Z_A') is obtained as

$$(Z_0', Z_A') = (Z_0, Z_B) \begin{pmatrix} v & 0 \\ (k_1^B - v\delta_1^B) + iv^B & k_A^B \end{pmatrix}, \tag{4.8}$$

where $v, v^B \in \mathbb{R}$ and the 3×3-matrix (k_A^B) satisfies

and also

$$(4.10) \qquad \det(k_B^A) = 1.$$

We denote by H' the subgroup of $GL(\mathbb{C}^4)$ consisting of all matrices of the form

$$(4.11) \qquad \mathfrak{M} = \begin{pmatrix} v & 0 \\ (k_1^B - v\delta_1^B) + iv^B & k_A^B \end{pmatrix},$$

which satisfy (4.9) and (4.10). We also denote by K the subgroup of $GL(\mathbb{R}^3)$ which consists of (k_B^A) satisfying (4.9) and (4.10). The group K is isomorphic to the Lorenz group $SO(2,1)$. Indeed, if O denote the orthogonal transformation for which $O^\tau \Lambda O$ equals the diagonal matrix with $1, 1, -1$ on the diagonal, then $K = O(SO(2,1))O^\tau$.

Note that (4.2), (4.4), and (4.5) imply that Z_0, considered as the affine point $z_0 \in \mathbb{C}^3$ where Z_0 and z_0 are as in (4.3), can be viewed as a point on $\Gamma_{\mathbb{C}}$. We denote by H'_0 the subgroup of H' consisting of those matrices which preserve Z_0 as an affine point on $\Gamma_{\mathbb{C}}$, i.e. the group of matrices of the form

$$(4.12) \qquad \mathfrak{N} = \begin{pmatrix} v & 0 \\ 0 & k_A^B \end{pmatrix},$$

where $(k_B^A) \in K$ with $k_1^A x_A = vx_1$. A straightforward calculation shows that (k_B^A) must be of the form

$$(4.13) \qquad (k_B^A) = \begin{pmatrix} v & a & \frac{1}{2}a^2 v^{-1} \\ 0 & 1 & av^{-1} \\ 0 & 0 & v^{-1} \end{pmatrix},$$

for some $a \in \mathbb{R}$. Thus, the group of frames for $\Gamma_{\mathbb{C}}$ (which, given a fixed frame, can be identified with the group H' via (4.8)) is a principal fiber bundle $P' \to \Gamma_{\mathbb{C}}$ with group H'_0. Let us now choose an orientation, as explained in §2, for the Levi nullbundle of $\Gamma_{\mathbb{C}}$ (which at a point Z_0 is spanned by x_1) and denote by P the group of frames consistent with this orientation. Then, as is easily verified, P is isomorphic to the group H, where H is the subgroup of H' consisting of matrices of the form (4.11) with $v > 0$. If we also denote by H_0 the subgroup of H'_0 consisting of those matrices of the form (4.12) for which $v > 0$, then $P \to M$ is a principal fiber bundle with group H_0. The reader should note that $\Gamma_{\mathbb{C}}$ has two connected components. This is reflected on the bundle P by the fact that the Lorenz group $SO(2,1)$ has two components.

A choice for the 4×4-matrix $\Pi = (\pi_b^a)$ of Maurer-Cartan forms for the group H is given by

$$(4.14) \qquad dZ_a = \pi_a^b Z_b.$$

The Maurer-Cartan equations of structure then become

$$(4.15) \qquad d\pi_a^b = \pi_a^c \wedge \pi_c^b,$$

which follows directly from differentiating (4.14). Due to the form (4.3) of the

the defining equations (4.5) and using again these equations, we deduce that the 3×3-matrix (π_A^B) (which is a Maurer-Cartan matrix for the group K) is given by

(4.16)
$$(\pi_A^B) = \begin{pmatrix} \pi_1^1 & \pi_2^1 & 0 \\ \pi_1^2 & 0 & \pi_2^1 \\ 0 & \pi_1^2 & -\pi_1^1 \end{pmatrix},$$

for some real 1-forms $\pi_1^1, \pi_2^1, \pi_1^2$. By differentiating $Z_0 = v(1, z_0)$, we obtain

(4.17)
$$dZ_0 = \frac{dv}{v}(v, vz_0) + v(0, dz_0) = \frac{dv}{v}Z_0 + (0, vdz_0).$$

Thus, by equation (4.5), we have

(4.18)
$$\frac{dv}{v}x_1 + \frac{1}{2}(vdz_0 + vd\bar{z}_0) = dx_1.$$

and hence, using also (4.14),

(4.19)
$$\frac{1}{2}(\pi_0^A + \bar{\pi}_0^A)x_A = \left(\pi_1^A - \frac{dv}{v}\delta_1^A\right)x_A.$$

Using (4.16), we obtain the equations

(4.20)
$$\begin{cases} \pi_1^1 - \pi_0^0 = \dfrac{1}{2}(\pi_0^1 + \bar{\pi}_0^1) \\[2mm] \pi_1^2 = \dfrac{1}{2}(\pi_0^2 + \bar{\pi}_0^2) \\[2mm] 0 = \pi_0^3 + \bar{\pi}_0^3. \end{cases}$$

The last formula in (4.20) implies that π_0^3 is a purely imaginary form. The matrix of 1-forms $\Pi = (\pi_b^a)$ is valued in the Lie algebra $\mathfrak{h}$ of H, and if we change frame using (4.8) then the corresponding matrix of 1-forms Π' for the new frame (Z_0', Z_A') is related to Π by

(4.21)
$$\Pi' = \operatorname{ad}(\mathfrak{M}^{-1})\Pi := \mathfrak{M}^{-1}\Pi\mathfrak{M},$$

where $\mathfrak{M}$ is the matrix given by (4.11). It follows that Π is a Cartan connection on P with group H (see e.g. [K, Chapter IV] and also [CM]), which is flat (i.e. with vanishing curvature form) by (4.15).

Let us relate the above to the results in previous sections. It is straightforward to verify that we can set

(4.22)
$$\begin{cases} \omega = -\dfrac{i}{2}\pi_0^3 \\[2mm] \omega^1 = \dfrac{1}{2i}\pi_0^2 \\[2mm] \theta^2 = \dfrac{1}{4i}\pi_0^1 \\[2mm] \Delta = \pi_0^0 + \pi_1^1 = 2\pi_0^0 + \dfrac{1}{2}(\pi_0^1 + \bar{\pi}_0^1) \\[2mm] \end{cases}$$

for the forms given by Theorem 3.1.37. Indeed, using (4.15) and (4.20), we obtain
the equations

$$
(4.23) \quad
\begin{cases}
d\omega = \Delta \wedge \omega + i\omega^{\bar{1}} \wedge \omega^{1} \\[4pt]
d\omega^{1} = \xi \wedge \omega + \dfrac{1}{2}\Delta \wedge \omega^{1} - i\omega^{1} \wedge \theta^{2} + i\theta^{\bar{2}} \wedge \omega^{1}) + \omega^{\bar{1}} \wedge \theta^{2} \\[4pt]
d\theta^{2} = \dfrac{1}{2}\xi \wedge \omega^{1} + 2i\theta^{\bar{2}} \wedge \theta^{2} \\[4pt]
d\Delta = i\xi \wedge \omega^{1} - i\xi \wedge \omega^{\bar{1}} \\[4pt]
d\xi = -\dfrac{1}{2}(\Delta + 2i\theta^{2} - 2i\theta^{\bar{2}}) \wedge \xi,
\end{cases}
$$

which satisfy the conditions of Theorem 3.1.37. Note that the invariant $\hat{k}$, defined
in section 3, satisfies $\hat{k} \equiv 2i$. Thus, in what follows, the phase function t and the
modulus r as defined by (3.1.2), are identically 1.

Recall that the group of frames is a principal fiber bundle $P \to \Gamma_{\mathbb{C}}$ with group
H_0. For any $\mathfrak{N} \in H_0$, where $\mathfrak{N} \in H_0$ is given by (4.12) and (k_B^A) by (4.13), a change
of frame $(Z_0', Z_A') = (Z_0, Z_B)\mathfrak{N}$ results in the change of connection form

$$
(4.24) \quad
\Pi' = \mathrm{ad}\,(\mathfrak{N}^{-1})\Pi =
\begin{pmatrix}
\pi_0^0 & 0 \\
vl_A^B \pi_0^A & l_A^B k_D^C \pi_C^A
\end{pmatrix},
$$

where (l_D^C) denotes the inverse of (k_B^A). Using (4.22) and calculating the inverse of
(k_A^B) given by (4.13), we deduce that (4.24) yields the corresponding transformation

$$
(4.25) \quad
\begin{cases}
\omega' = v^2\omega \\[4pt]
(\omega^1)' = -av\omega + v\omega^1 \\[4pt]
(\theta^2)' = \dfrac{1}{4}a^2\omega - \dfrac{1}{2}a\omega^1 + \theta^2 \\[4pt]
\Delta' = iav^{-1}(\omega^{\bar{1}} - \omega^1) + \Delta \\[4pt]
\xi' = \dfrac{1}{2}ia^2 v^{-1}(\omega^1 - \omega^{\bar{1}}) - iav^{-1}(\theta^2 - \theta^{\bar{2}}) - \dfrac{1}{2}av^{-1}\Delta + v^{-1}\xi.
\end{cases}
$$

In particular, we obtain

$$
(4.26) \quad
\begin{pmatrix}
\omega' \\
(\omega^1)' \\
(\omega^{\bar{1}})'
\end{pmatrix}
=
\begin{pmatrix}
u & 0 & 0 \\
x & \sqrt{u} & 0 \\
x & 0 & \sqrt{u}
\end{pmatrix}
\begin{pmatrix}
\omega \\
\omega^1 \\
\omega^{\bar{1}}
\end{pmatrix},
$$

where

$$
(4.27) \quad u = v^2, \quad x = -av.
$$

Hence, we have defined an isomorphism $\phi\colon H_0 \to G_2$, where G_2 is as defined in
section 3.1 with $t \equiv 1$, defined by

$$
(4.28) \quad
\phi
\begin{pmatrix}
v & 0 & 0 & 0 \\
0 & v & a & \frac{1}{2}a^2 v^{-1} \\
0 & 0 & 1 & av^{-1}
\end{pmatrix}
=
\begin{pmatrix}
v^2 & 0 & 0 \\
-av & v & 0 \\
-av & 0 & v
\end{pmatrix}.
$$

Let M be any 5-dimensional CR manifold of hypersurface type which satisfies the conditions in Theorem 3.1.37. Furthermore, we assume that the invariant $\hat{k} \equiv 2i$ in a neighborhood of p_0. Let $\omega, \omega^1, \theta^2, \Delta, \xi$ be the forms given by Theorem 3.1.37 such that $\underline{\omega}$ is given by (3.1.36). Define the $\mathfrak{h}$-valued 1-form $\Pi = (\pi_a^b)$ by

$$(4.29) \qquad \Pi := \begin{pmatrix} \pi_0^0 & 0 & 0 & 0 \\ \pi_0^1 & \pi_1^1 & \pi_2^1 & 0 \\ \pi_0^2 & \pi_1^2 & 0 & \pi_2^1 \\ \pi_0^3 & 0 & \pi_1^2 & -\pi_1^1 \end{pmatrix},$$

where

$$\pi_0^0, \pi_0^1, \pi_0^2, \pi_0^3, \pi_1^1, \pi_1^2, \pi_2^1,$$

are obtained by solving (4.22), using also the first two equation of (4.20). Clearly, Π defines an isomorphism between $T_y Y_2$, where $Y_2 \to M$ is the principal bundle given by Theorem 3.1.37, and $\mathfrak{h}$ for every $y \in Y_2$. However, it is not difficult to verify that Π is in general not a Cartan connection, i.e. it does not transform according to (4.21). Nevertheless, by defining the curvature

$$(4.30) \qquad \Omega = d\Pi - \Pi \wedge \Pi,$$

a direct consequence of Cartan's solution of the equivalence problem for $\{1\}$- structures (see e.g. [G]) is the following characterization of the tube over the light cone.

Theorem 4.31. *Let M be a 5-dimensional real-analytic CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Assume that $\hat{k} \equiv 2i$. Choose an orientation for the Levi nullspace $\mathfrak{N}_{p_0}$ (as explained in §2), and denote by $Y_2 \to M$ the principal bundle with 1-form $\underline{\omega}$ given by Theorem 3.1.37. Then, the $\mathfrak{h}$-valued 1-form Π defined by (4.29), where*

$$(4.32) \qquad \begin{cases} \pi_0^0 := \dfrac{1}{2}(\Delta - 2i\theta^2 + 2i\theta^{\bar{2}}) \\[2mm] \pi_1^1 := \dfrac{1}{2}(\Delta + 2i\theta^2 - 2i\theta^{\bar{2}}) \\[2mm] \pi_0^1 := 4i\theta^2 \\[2mm] \pi_0^2 := 2i\omega^1 \\[2mm] \pi_0^3 := 2i\omega \\[2mm] \pi_1^2 := i(\omega^1 - \omega^{\bar{1}}) \\[2mm] \pi_2^1 := -\xi \end{cases}$$

and $\underline{\omega}$ is given by (3.1.36), defines an isomorphism $T_y Y_2 \cong \mathfrak{h}$ for every $y \in Y_2$, with the following property. There exists a local real-analytic CR diffeomorphism $f \colon M \to \Gamma_{\mathbb{C}}$ near p_0 if and only if the curvature Ω given by (4.30) vanishes identically.

We conclude the discussion of the tube over the light cone by computing the dimension of the stability group $\mathrm{Aut}(\Gamma_{\mathbb{C}}, p_0)$ of $\Gamma_{\mathbb{C}}$ at a point $p_0 \in \Gamma_{\mathbb{C}}$. Observe that, given a frame (Z_0, Z_A) in P, the manifold $\Gamma_{\mathbb{C}}$ can be viewed as the quotient

point on $\Gamma_{\mathbb{C}}$ corresponding to Z_0 by p_0. Then, under the identification $\Gamma_{\mathbb{C}} \cong H/H_0$, p_0 corresponds to the coset eH_0, where $e \in H$ denotes the identity matrix. The group H_0 acts on the left on H/H_0 and each homomorphism $aH_0 \mapsto baH_0$, for $b \in H_0$, preserves the point $p_0 \cong eH_0$. It is straightforward to verify that the action is effective; i.e. if, for $b \in H_0$, the homomorphism $aH_0 \mapsto baH_0$ is the identity, then $b = e$. Let us denote by $f_b \colon (\Gamma_{\mathbb{C}}, p_0) \to (\Gamma_{\mathbb{C}}, p_0)$ the mapping corresponding to the homomorphism $aH_0 \mapsto baH_0$. Each f_b is a CR diffeomorphism. (Indeed, it is not difficult to compute f_b in coordinates and see that f_b is induced by an invertible linear transformation of $\mathbb{C}^3$.) Thus, $b \mapsto f_b$ embeds H_0 as a subgroup of $\mathrm{Aut}(\Gamma_{\mathbb{C}}, p_0)$. Since $\dim H_0 = 2$, we conclude that $\dim \mathrm{Aut}(\Gamma_{\mathbb{C}}, p_0) \geq 2$. On the other hand, by Theorem 3.1.37 and [K, Theorem 3.2], it follows (as in the introduction) that the subgroup of $\mathrm{Aut}(\Gamma_{\mathbb{C}}, p_0)$ consisting of those CR diffeomorphisms that preserve the orientation of the Levi nullspace chosen above embeds as a closed submanifold of $P_{p_0} \cong H_0$. Hence, we have $\dim \mathrm{Aut}(\Gamma_{\mathbb{C}}, p_0) = 2$.

5. Concluding remarks; the higher dimensional case

Let us briefly return to the situation in section 2, i.e. M is a smooth CR manifold (of hypersurface type and dimension $2n+1$) which is Levi uniform of rank $n-1$ at p_0. We also assume that M satisfies Condition 2.21 and 2.25 (which in particular imply that M is pseudoconvex and 2-nondegenerate at p_0). As in section 3, we consider equation (2.35) which, in view of (2.13), can be rewritten as follows

$$(5.1) \qquad \tilde{k}_{\bar{\mu}} = \hat{k}_{\bar{\mu}} + 2i\hat{g}^{\bar{\alpha}\nu}c_{\nu}\hat{h}_{\bar{\alpha}\bar{\mu}} - \overline{c_{\mu}}\hat{k}.$$

Now, using (2.29) we deduce that

$$(5.2) \qquad \tilde{k}_{\bar{\mu}} = \hat{k}_{\bar{\mu}} + 2ic_{\mu}\lambda_{\bar{\mu}} - \overline{c_{\mu}}\hat{k}.$$

Thus, either $|\hat{k}(p_0)| \neq 2\lambda_{\bar{\mu}}$ for $\mu = 1, 2, \ldots, n-1$, or $|\hat{k}(p_0)| \neq 2\lambda_{\bar{\mu}_0}$ for some $\mu_0 \in \{1, 2, \ldots, n-1\}$. In the first case, we can solve for each c_{μ} in the equation $\tilde{k}_{\bar{\mu}} = 0$ and proceed as in section 3.2 to construct a principal bundle $P \to M$ with 1-form $\underline{\omega}$ reducing the CR structure on M to a parallelism. In the latter case, we can solve for $c_{\bar{\mu}}$ in the equation $\tilde{k}_{\bar{\mu}} = 0$ for all $\mu \neq \mu_0$. We then proceed as in in section 3.1 to determine $c_{\bar{\mu}_0}$ and construct the bundle $P \to M$ with 1-form $\underline{\omega}$. We do not give the details here. Conditions 2.21 and 2.25 do not appear to be natural when $n \geq 3$. (Recall, however, that these two conditions reduce to 2-nondegeneracy and Levi uniformity when $n = 2$.) In particular, the tube over the light cone in $\mathbb{C}^{n+1}$, $n \geq 3$, does not satisfy these conditions.

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